TOWARDS ASYMPTOTIC COMPLETENESS OF TWO-PARTICLE SCATTERING IN LOCAL RELATIVISTIC QFT

WOJCIECH DYBALSKI AND CHRISTIAN GÉRARD

Abstract. We consider the problem of existence of asymptotic observables in local relativistic theories of massive particles. Let $\tilde{p}_1$ and $\tilde{p}_2$ be two energy-momentum vectors of a massive particle and let Δ be a small neighbourhood of $\tilde{p}_1 + \tilde{p}_2$. We construct asymptotic observables (two-particle Araki-Haag detectors), sensitive to neutral particles of energy-momenta in small neighbourhoods of $\tilde{p}_1$ and $\tilde{p}_2$. We show that these asymptotic observables exist, as strong limits of their approximating sequences, on all physical states from the spectral subspace of Δ. Moreover, the linear span of the ranges of all such asymptotic observables coincides with the subspace of two-particle Haag-Ruelle scattering states with total energy-momenta in Δ. The result holds under very general conditions which are satisfied, for example, in $\lambda\phi_2^4$. The proof of convergence relies on a variant of the phase-space propagation estimate of Graf.

1. Introduction

The question of a complete particle interpretation of quantum theories is of fundamental importance for our understanding of physics. The solution of this problem in non-relativistic quantum mechanics, obtained in [En78, SiSo87, Gr90, De93] for a large class of physically relevant Hamiltonians, requires the convergence of suitably chosen time-dependent families of observables. The existence of these limits, called *asymptotic observables*, relies on the method of *propagation estimates* [SiSo87, Gr90], which is a refined variant of the Cook method. This technique was later adapted to non-relativistic QFT in [DG99] which initiated a systematic study of the problem of asymptotic completeness in this context [DG00, FGS02, FGS04, DM12]. In the present work we implement the method of propagation estimates in local relativistic quantum field theories of massive particles. We obtain the existence of certain asymptotic observables which can be interpreted as two-particle detectors. Our results, stated in Theorems 2.6 and 2.7 below, hold in any massive theory satisfying the Haag-Kastler axioms, for example in $\lambda\phi_2^4$. Our work sheds a new light on the problem of asymptotic completeness in such theories, which is widely open to date.

The problem of existence of asymptotic observables in the framework of algebraic quantum field theory (cf. Subsection 2.1) was first studied in the seminal work of Araki and Haag [AH67] and later by Enss in [En75]. These authors considered families of observables of the form

$$C_t := \int h\left(\frac{x}{t}\right) C(t,x)dx = \mathrm{e}^{\mathrm{i}tH} \int h\left(\frac{x}{t}\right) C(x)dx\, \mathrm{e}^{-\mathrm{i}tH}, \tag{1.1}$$

where C denotes a suitable (almost local) observable, $C(t,x)$ its translation in space-time by $(t,x) \in \mathbb{R}^{1+d}$, H is the full Hamiltonian of the relativistic theory and $h \in C_0^\infty(\mathbb{R}^d)$. They were able to show that products of such observables

$$Q_{n,t} = C_{1,t} \dots C_{n,t}, \tag{1.2}$$

associated with functions h_i, $i = 1, \dots, n$, with mutually disjoint supports, converge, as $t \to +\infty$, on suitably chosen *domains of Haag-Ruelle scattering states*[1] (cf. Section 6). The limit Q_n^+ can be interpreted as a coincidence arrangement of detectors which is sensitive to states containing a configuration of n particles, with velocities in the supports of the functions $h_1, \dots, h_n$.

1991 *Mathematics Subject Classification.* 81T05, 81U99.

Key words and phrases. local quantum field theory, Haag-Ruelle scattering theory, Araki-Haag detectors, asymptotic completeness.

[1]We consider only the limit $t \to +\infty$ and outgoing scattering states in this paper as the case $t \to -\infty$ is completely analogous.

An important advance was made by Buchholz, who proved, for a sufficiently large class of observables C, the following bound:

$$\sup_{t\in\mathbb{R}} \|C_t 1\!\!1_\Delta(U)\| < \infty, \tag{1.3}$$

where $1\!\!1_\Delta(U)$ is the projection on states whose energy-momentum belongs to a bounded Borel set Δ. (See [Bu90] and Lemma 3.3 below). This a priori estimate is a foundation of the theory of particle weights [BPS91, Po04a, Po04b, Dy10, DT11b, DT11a] and it implies, in particular, that the sequences $Q_{n,t}$ converge on *all Haag-Ruelle scattering states of bounded energy.* However, the question of their convergence on the orthogonal complement of the subspace of scattering states, which is of crucial importance for the problem of a complete particle interpretation of the theory (cf. Chapter 6 of [Ha]), remained unanswered to date.

In this paper we give a solution of this problem in the case of $n = 2$ for Araki-Haag detectors (1.2) sensitive to massive neutral particles. More precisely, let $\tilde{p}_1, \tilde{p}_2$ be two energy-momentum vectors of massive particles. We choose almost local observables B_1, B_2 whose energy-momentum transfers belong to small neighbourhoods of $-\tilde{p}_1$, $-\tilde{p}_2$, respectively, and set $C_1 := B_1^*B_1$, $C_2 := B_2^*B_2$. Now let Δ be a small neighbourhood of $\tilde{p}_1 + \tilde{p}_2$. Our main result is the existence of

$$Q_2^+(\Delta) := \mathrm{s-}\lim_{t\to+\infty} C_{1,t}C_{2,t}1\!\!1_\Delta(U). \tag{1.4}$$

Moreover, we show that the union of the ranges of all the operators $Q_2^+(\Delta)$, constructed as above, coincides with the subspace of two-particle Haag-Ruelle scattering states, whose total energy-momenta belong to Δ. This latter result, stated precisely in Thm. 2.7 below, can be interpreted as a weak variant of two-particle asymptotic completeness. We point out that this generalized concept of complete particle interpretation does not imply the conventional one.

To illustrate this point, let us give a simple example of a theory which satisfies our general assumptions from Subsect. 2.1 and is not asymptotically complete in the conventional sense: Let $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ be the net of local algebras of massive scalar free field theory acting on the Fock space $\mathcal{F}$ and let U be the corresponding unitary representation of translations. Let $\mathcal{O} \mapsto \mathfrak{A}_{\mathrm{ev}}(\mathcal{O})$ be a subnet generated by even functions of the fields acting on the subspace $\mathcal{F}_{\mathrm{ev}} \subset \mathcal{F}$ spanned by vectors with even particle numbers and let us set $U_{\mathrm{ev}} = U|_{\mathcal{F}_{\mathrm{ev}}}$. Then the net $\hat{\mathfrak{A}}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}) \otimes \mathfrak{A}_{\mathrm{ev}}(\mathcal{O})$, acting on $\mathcal{F} \otimes \mathcal{F}_{\mathrm{ev}}$ and equipped with the unitary representation of translations $\widehat{U} = U \otimes U_{\mathrm{ev}}$, satisfies the assumptions from Subsect. 2.1 but is not asymptotically complete in the conventional sense. In fact, the subspace $\Omega \otimes \mathcal{F}_{\mathrm{ev}}$, where Ω is the Fock space vacuum, is orthogonal to all the Haag-Ruelle scattering states of the theory (except for the vacuum). In physical terms, this subspace describes 'pairs of oppositely charged particles', whose mass hyperboloids do not appear in the vacuum sector. Due to the choice of the energy-momentum transfers of B_i, the asymptotic observables $Q_2^+(\Delta)$ annihilate such pairs of charged particles and, as stated in Thm. 2.7 below, only neutral particles remain in their ranges.

We would like to stress that our result applies to concrete interacting quantum field theories, as for example the $\lambda\phi_2^4$ model. This theory is known to possess a lower and upper mass gap at small coupling constants λ, but its particle aspects are rather poorly understood. Asymptotic completeness is only known for total energies from the intervals $[0, 3m-\varepsilon]$ and $[3m+\varepsilon, 4m-\varepsilon]$, where m is the particle mass and $\varepsilon \to 0$ as $\lambda \to 0$ [GJS73, SZ76, CD82]. Since we can choose the region Δ in (1.4) outside of these intervals, our result provides new information about the asymptotic dynamics of this theory.

Let us now describe briefly the main ingredients of the proof of existence of the limit (1.4): Let $Q_{2,t}(\Delta)$ be the approximants on the r.h.s. of (1.4). Exploiting locality and the disjointness of supports of h_1, h_2 one can write

$$\begin{aligned} Q_{2,t}(\Delta) = \int h_1\left(\tfrac{x_1}{t}\right) h_2\left(\tfrac{x_2}{t}\right) B_1^*(t,x_1)B_2^*(t,x_2)B_1(t,x_1)B_2(t,x_2)1\!\!1_\Delta(U)dx_1dx_2 \\ +O(t^{-\infty}), \end{aligned} \tag{1.5}$$

where $O(t^{-\infty})$ is a term tending to zero in norm faster than any inverse power of t. In the next step we exploit our assumptions on the energy-momentum transfers of B_1, B_2, which give for

any $\Psi \in \mathrm{Ran}\, 1\!\!1_{\Delta}(U)$:

$$B_1(t,x_1)B_2(t,x_2)\Psi = \Omega(\Omega|B_1(t,x_1)B_2(t,x_2)\Psi), \tag{1.6}$$

due to the presence of the lower mass-gap. Thus we obtain

$$Q_{2,t}(\Delta)\Psi = \int \mathrm{H}_t(x_1,x_2)F_t(x_1,x_2)B_1^*(t,x_1)B_2^*(t,x_2)\Omega dx_1 dx_2 + O(t^{-\infty}), \tag{1.7}$$

where

$$F_t(x_1,x_2) := (\Omega|B_1(t,x_1)B_2(t,x_2)\Psi),\ \mathrm{H}_t(x_1,x_2) := h_1\left(\frac{x_1}{t}\right)h_2\left(\frac{x_2}{t}\right).$$

We note that by replacing $\mathrm{H}_t(x_1,x_2)F_t(x_1,x_2)$ in the first term on the r.h.s. of (1.7) with $g_1(t,x_1)g_2(t,x_2)$, where g_1, g_2 are positive energy solutions of the Klein-Gordon equation, one would obtain a Haag-Ruelle scattering state (cf. Thm. 6.5). While such replacement is not possible at finite times, it turns out that it can be performed asymptotically. In fact, Thm. 4.1 below, reduces the problem of strong convergence of $t \mapsto Q_{2,t}(\Delta)$ to the existence of the following limit in the norm topology of $L^2(\mathbb{R}^{2d})$:

$$F_+ := \lim_{t\to\infty} \mathrm{e}^{\mathrm{i}t\tilde{\omega}(D_{\tilde{x}})}\mathrm{H}_t F_t, \tag{1.8}$$

where $\tilde{x} = (x_1,x_2) \in \mathbb{R}^{2d}$, $\tilde{\omega}(D_{\tilde{x}}) = \omega(D_{x_1}) + \omega(D_{x_2})$ and $\omega(k) = \sqrt{k^2+m^2}$ is the dispersion relation of the massive particles under study.

A large part of our paper is devoted to the proof of existence of the limit (1.8). In the first step, taken in Lemma 4.2, we show that F_t satisfies the following inhomogeneous evolution equation

$$\partial_t F_t = -\mathrm{i}\tilde{\omega}(D_{\tilde{x}})F_t + \langle R\rangle_t, \tag{1.9}$$

where, using locality, we show that the term $\langle R\rangle_t$ satisfies $\|\tilde{\mathrm{H}}_t\langle R\rangle_t\|_2 = O(t^{-\infty})$, for any $\tilde{\mathrm{H}}_t(\tilde{x}) := \tilde{\mathrm{H}}\left(\frac{\tilde{x}}{t}\right)$ with $\tilde{\mathrm{H}} \in C_0^\infty(\mathbb{R}^{2d})$ vanishing near the diagonal $\{x_1 = x_2\}$. Given (1.9), we prove the existence of the limit (1.8) by extending the method of propagation estimates to inhomogeneous evolution equations.

An important step is to obtain a *large velocity estimate*, for which the usual quantum mechanical proof does not apply, since in our case all propagation observables must vanish near the diagonal. Instead we use a relativistic argument, based on the fact that hyperplanes $\{t = v\cdot x\}$ for $|v| > 1$ are space-like (see Lemma 5.1). Another key ingredient is a *phase-space propagation estimate*, whose proof follows closely the usual quantum mechanical one. One new aspect, to which we will come back below, is the fact that the convex Graf function R must now vanish near the diagonal. By combining the two propagation estimates in Prop 5.5, we obtain the existence of the limit (1.8) and therefore the convergence of Araki-Haag detectors (1.4).

It is a natural question if the convergence of $Q_{n,t}$ can also be shown for $n \neq 2$ by the methods described above. Perhaps surprisingly, this does not seem to be the case for $n = 1$, since it is difficult to filter out possible 'pairs of charged particles' using only one detector (cf. the discussion above). However, the situation looks much better for $n > 2$. Here the initial steps of our analysis can be carried out and difficulties arise only at the level of the phase-space propagation estimate: The Graf function R must vanish not only near the diagonal $x_1 = x_2$, but also near all the other collision planes $x_1 = x_3$, $x_2 = x_3$ etc. Since R is supposed to be convex in some ball around the origin it must be zero in a neighbourhood of the convex hull of the collision planes restricted to this ball. Thus a large and physically interesting part of the configuration space is out of reach of the phase-space propagation estimate for $n > 2$. It seems to us that new propagation estimates have to be developed to handle this problem.

We would like to point out that our analysis is closely related to quantum-mechanical scattering theory for dispersive systems (see e.g. [Ge91, Zi97]). A simple example of a dispersive system is the following Hamiltonian

$$H_{\mathrm{d}} = \sum_{i=1}^{n}\omega(D_{x_i}) + \sum_{i<j}V(x_i - x_j), \tag{1.10}$$

where $V \in \mathcal{S}(\mathbb{R}^d)$. We note that the corresponding Schrödinger equation has the form

$$\partial_t \Psi_t = -\mathrm{i} \sum_{i=1}^{n} \omega(D_{x_i})\Psi_t - \mathrm{i} \sum_{i<j} V(x_i - x_j)\Psi_t, \tag{1.11}$$

where $\Psi_t = \mathrm{e}^{-\mathrm{i}tH_\mathrm{d}}\Psi$, $\Psi \in L^2_{\mathrm{sym}}((\mathbb{R}^d)^{\times n})$. For $n = 2$ equation (1.11) has a form of the evolution equation (1.9) with $F_t = \Psi_t$, and $\langle R \rangle_t = -\mathrm{i}V(x_1 - x_2)\Psi_t$ which satisfies $\|G_t \langle R \rangle_t\|_2 = O(t^{-\infty})$ as a consequence of the rapid decay of the potential. In the light of our discussion of equation (1.9), it is not a surprise that asymptotic completeness holds for dispersive systems for $n = 2$, (which is actually a well known fact). However, the case $n > 2$ is still open and requires new ideas.

Our paper is organized as follows: In Sect. 2 we recall the framework of local relativistic quantum field theory and state precisely our results. In Sect. 3 we introduce some notation and terminology and collect the main properties of particle detectors. In Sect. 4 we reduce the problem of convergence of the families of observables (1.4) to the existence of the limit (1.8) and derive the inhomogeneous evolution equation (1.9). In Sect. 5 we prove the convergence in (1.8) by showing large velocity and phase-space propagation estimates. In Sect. 6 we recall some basic facts on the Haag-Ruelle scattering theory in the two-particle case. The proof of Thm. 2.7, which gives a weak form of two-particle asymptotic completeness, is presented in Sect. 7. In Appendix A we state some generalizations of standard abstract arguments to the inhomogeneous evolution equations. They are used in Section 5.

Acknowledgment: W.D. would like to thank Detlev Buchholz for pointing out to him the problem of existence of particle detectors in relativistic QFT and numerous interesting discussions. W.D. is also grateful for stimulating discussions with Jacob Schach Møller, Alessandro Pizzo and Wojciech De Roeck concerning the problem of existence of asymptotic observables in non-relativistic QFT. W.D. acknowledges financial support of the German Research Foundation (DFG) within the stipend DY107/1–1 and hospitality of the Hausdorff Research Institute for Mathematics, Bonn.

2. Framework and results

In this section we recall the conventional framework of local quantum field theory and formulate precisely our main results.

2.1. **Nets of local observables.** As usual in the Haag-Kastler framework of local quantum field theory, we consider a net

$$\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \subset B(\mathcal{H})$$

of von Neumann algebras attached to open bounded regions of Minkowski space-time $\mathbb{R}^{1+d}$, which satisfies the assumptions of isotony, locality, covariance w.r.t. translations, positivity of energy, uniqueness of the vacuum and cyclicity of the vacuum.

The assumption of *isotony* says that $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$ if $\mathcal{O}_1 \subset \mathcal{O}_2$. It allows to define the C^*-inductive limit of the net, which will be denoted by $\mathfrak{A}$. *Locality* means that $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)'$ if $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated. To formulate the remaining postulates, we assume that there exists a strongly continuous unitary representation of translations

$$\mathbb{R}^{1+d} \ni (t, x) \mapsto U(t, x) =: \mathrm{e}^{\mathrm{i}(tH - x\cdot P)} \text{ on } \mathcal{H}.$$

We also introduce the group of automorphisms of $\mathfrak{A}$ induced by U:

$$\alpha_{t,x}(B) := B(t, x) := U(t, x)BU^*(t, x),\ B \in \mathfrak{A},\ (t, x) \in \mathbb{R}^{1+d}.$$

The assumption of *covariance* says that

$$\alpha_{t,x}(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O} + (t, x)),\ \forall \text{ open bounded } \mathcal{O} \text{ and } (t, x) \in \mathbb{R}^{1+d}. \tag{2.1}$$

We will need a restrictive formulation of positivity of energy, suitable for massive theories. We denote by $H_m := \{(E, p) \in \mathbb{R}^{1+d}\ :\ E = \sqrt{p^2 + m^2}\}$ the mass hyperboloid of a particle of mass

$m > 0$ and set $G_\mu := \{(E,p) \in \mathbb{R}^{1+d} \ : \ E \geq \sqrt{p^2+\mu^2}\}$. We assume that:

$$\begin{aligned} &i) \quad \mathcal{S}p\, U = \{0\} \cup H_m \cup G_\mu \text{ for some } m < \mu \leq 2m, \\ &ii) \quad 1\!\!1_{\{0\}}(U) = |\Omega\rangle\langle\Omega|, \ \Omega \text{ cyclic for } \mathfrak{A}. \end{aligned} \tag{2.2}$$

The unit vector Ω will be called the *vacuum vector*, we denoted by $\mathcal{S}p\, U \subset \mathbb{R}^{1+d}$ the spectrum of (H,P) and by $1\!\!1_\Delta(U)$ the spectral projection on a Borel set $\Delta \subset \mathbb{R}^{d+1}$. Part $i)$ in (2.2) encodes *positivity of energy* and the presence of an upper and lower mass-gap. Part $ii)$ covers the *uniqueness and cyclicity of the vacuum.*

2.2. **Relevant classes of observables.** In this subsection we introduce some classes of observables, which enter into the formulation of our main results. First, we recall the definition of *almost local* operators.

Definition 2.1. *$B \in \mathfrak{A}$ is* almost local *if there exists a family $A_r \in \mathfrak{A}(\mathcal{O}_r)$, where $\mathcal{O}_r := \{x \in \mathbb{R}^{1+d} \ : \ |x| \leq r\}$ is the double cone of radius r centered at 0, s.t. $\|B - A_r\| \in O(\langle r\rangle^{-\infty})$.*

To introduce another important class – the *energy-decreasing* operators – we need some definitions: If $B \in \mathfrak{A}$, we denote by $\widehat{B}$ its Fourier transform:

$$\widehat{B}(E,p) := (2\pi)^{-(1+d)/2} \int \mathrm{e}^{-\mathrm{i}(Et - p\cdot x)} B(t,x) dt dx, \tag{2.3}$$

defined as an operator-valued distribution. We denote by $\mathrm{supp}(\widehat{B}) \subset \mathbb{R}^{1+d}$ the support of $\widehat{B}$, called the *energy-momentum transfer of B*. We recall the following well-known properties:

$$\begin{aligned} &i) \quad \widehat{\alpha_{t,x}(B)}(E,p) = \mathrm{e}^{\mathrm{i}(Et - p\cdot x)} \widehat{B}(E,p), \\ &ii) \quad \mathrm{supp}(\widehat{B^*}) = -\mathrm{supp}(\widehat{B}), \\ &iii) \quad B 1\!\!1_\Delta(U) = 1\!\!1_{\overline{\Delta + \mathrm{supp}(\widehat{B})}}(U) B 1\!\!1_\Delta(U), \ \forall \text{ Borel sets } \Delta \subset \mathbb{R}^{1+d}. \end{aligned} \tag{2.4}$$

Now we are ready to define the energy-decreasing operators:

Definition 2.2. *$B \in \mathfrak{A}$ is* energy decreasing *if* $\mathrm{supp}(\widehat{B}) \cap V_+ = \emptyset$*, where $V_+ := \{(E,p) \ : \ E \geq |p|\}$ is the closed forward light cone.*

In the rest of the paper we will work with the following set of observables:

Definition 2.3. *We denote by $\mathcal{L}_0 \subset \mathfrak{A}$ the subspace spanned by $B \in \mathfrak{A}$ such that:*

$$\begin{aligned} &i) \quad B \textit{ is energy decreasing}, \ \mathrm{supp}(\widehat{B}) \textit{ is compact}, \\ &ii) \quad \mathbb{R}^{1+d} \ni (t,x) \mapsto B(t,x) \in \mathfrak{A} \textit{ is } C^\infty \textit{ in norm}, \\ &iii) \quad \partial^\alpha_{t,x} B(t,x) \textit{ is almost local for all } \alpha \in \mathbb{N}^{1+d}. \end{aligned}$$

Note that if *i)* and *ii)* hold, then $\partial^\alpha_{t,x} B(t,x)$ is energy decreasing for any $\alpha \in \mathbb{N}^{1+d}$. Note also that if $A \in \mathfrak{A}(\mathcal{O})$ and $f \in \mathcal{S}(\mathbb{R}^{1+d})$ with $\mathrm{supp}\widehat{f}$ compact and $\mathrm{supp}\widehat{f} \cap V_+ = \emptyset$ then

$$B = (2\pi)^{-(1+d)/2} \int f(t,x) A(t,x) dt dx \tag{2.5}$$

belongs to $\mathcal{L}_0$ by (2.4) *i)*, since $\widehat{B}(E,p) = \widehat{f}(E,p)\widehat{A}(E,p)$. (See (3.1) below for definition of $\widehat{f}$).

2.3. **Results.** For any $B_1, B_2 \in \mathcal{L}_0$ and $h_1, h_2 \in C_0^\infty(\mathbb{R}^d)$ with disjoint supports we define the approximating families of *one-particle detectors*:

$$C_{1,t} := \int h_1\left(\frac{x_1}{t}\right) (B_1^* B_1)(t, x_1) dx_1, \quad C_{2,t} := \int h_2\left(\frac{x_2}{t}\right) (B_2^* B_2)(t, x_2) dx_2 \tag{2.6}$$

which have appeared already in (1.1) above. We note that in view of Lemma 3.3, stated below, $\sup_{t\in\mathbb{R}} \|C_{i,t} 1\!\!1_{\tilde{\Delta}}(U)\| < \infty$, $i = 1, 2$, for any bounded Borel set $\tilde{\Delta}$.

Now for any open bounded subset $\Delta \subset G_{2m}$ we define the *two-particle detectors*:

$$Q_{2,t}(\Delta) := C_{1,t} C_{2,t} 1\!\!1_\Delta(U). \tag{2.7}$$

Our main result is the strong convergence of $Q_{2,t}(\Delta)$ as $t \to \infty$ if the extension of Δ is smaller than the mass-gap (i.e., $(\overline{\Delta} - \overline{\Delta}) \cap \mathcal{S}p\, U = \{0\}$) and (B_1, B_2) is $\Delta-$admissible in the following sense:

Definition 2.4. *Let $\Delta \subset \mathbb{R}^{1+d}$ be an open bounded set and $B_1, B_2 \in \mathcal{L}_0$. We say that (B_1, B_2) is $\Delta-$*admissible *if*

$$(-\mathrm{supp}(\widehat{B}_i)) \cap \mathcal{S}p\, U \subset H_m, \ \ i = 1, 2, \tag{2.8}$$

$$-\left(\mathrm{supp}(\widehat{B}_1) + \mathrm{supp}(\widehat{B}_2)\right) \subset \Delta, \tag{2.9}$$

$$(\overline{\Delta} + \mathrm{supp}(\widehat{B}_1) + \mathrm{supp}(\widehat{B}_2)) \cap \mathcal{S}p\, U \subset \{0\}. \tag{2.10}$$

Remark 2.5. *In Lemma 7.4, it is shown that if $\Delta \subset G_{2m}$ is an open bounded set s.t. $(\overline{\Delta} - \overline{\Delta}) \cap \mathcal{S}p\, U \subset \{0\}$ and $-\mathrm{supp}(\widehat{B}_1)$, $-\mathrm{supp}(\widehat{B}_2)$ are sufficiently small neighbourhoods of vectors $\tilde{p}_1, \tilde{p}_2 \in H_m$ s.t. $\tilde{p}_1 \neq \tilde{p}_2$ and $\tilde{p}_1 + \tilde{p}_2 \in \Delta$ then (B_1, B_2) is $\Delta-$admissible.*

Theorem 2.6. *Let $\Delta \subset G_{2m}$ be an open bounded set such that $(\overline{\Delta} - \overline{\Delta}) \cap \mathcal{S}p\, U = \{0\}$. Let $B_1, B_2 \in \mathcal{L}_0$ be $\Delta-$admissible and suppose that $h_1, h_2 \in C_0^\infty(\mathbb{R}^d)$ have disjoint supports. Then there exists the limit*

$$Q_2^+(\Delta) := \mathrm{s-}\lim_{t\to\infty} C_{1,t} C_{2,t} 1\!\!1_\Delta(U), \tag{2.11}$$

where $C_{i,t}$ are defined in (2.6) for B_i, h_i, $i = 1, 2$. The range of $Q_2^+(\Delta)$ belongs to $1\!\!1_\Delta(U)\mathcal{H}_2^+$, where $\mathcal{H}_2^+$ is the subspace of two-particle Haag-Ruelle scattering states defined in Thm. 6.5.

Proof. Follows immediately from Theorems 4.1 and 5.5. □

Thm. 2.6 is complemented by Thm. 2.7, stated below, which says that any two-particle scattering state can be prepared with the help of Araki-Haag detectors. This weak variant of two-particle asymptotic completeness ensures, in particular, that sufficiently many asymptotic observables (2.11) are non-zero. The proof is given in Sect. 7.

Theorem 2.7. *Let $\Delta \subset G_{2m}$ be an open bounded set such that $(\overline{\Delta} - \overline{\Delta}) \cap \mathcal{S}p\, U = \{0\}$. Let J be the collection of quadruples $\alpha = (B_1, B_2, h_1, h_2)$ satisfying the conditions from Thm. 2.6 and let $Q_{2,\alpha}^+(\Delta)$ be the limit (2.11) corresponding to α. Then*

$$1\!\!1_\Delta(U)\mathcal{H}_2^+ = \mathrm{Span}\{\mathrm{Ran}\, Q_{2,\alpha}^+(\Delta) \ : \ \alpha \in J\}^{\mathrm{cl}}. \tag{2.12}$$

3. Preparations

In this section we introduce some notation and collect some properties of particle detectors.

3.1. **Notation.**

- By x, x_1, x_2 we denote elements of $\mathbb{R}^d$. We set $\tilde{x} = (x_1, x_2)$ to denote elements of $\mathbb{R}^{2d}$.
- we write $K \Subset \mathbb{R}^{1+d}$ if K is a compact subset of $\mathbb{R}^{1+d}$.
- we set $\langle x \rangle := (1 + x^2)^{\frac{1}{2}}$ for $x \in \mathbb{R}^d$ and $\omega(p) = (p^2 + m^2)^{\frac{1}{2}}$ for $p \in \mathbb{R}^d$.
- the momentum operator $\mathrm{i}^{-1}\nabla_x$ will be denoted by D_x.
- we denote by (t, x) or (E, p) the elements of $\mathbb{R}^{1+d}$.
- if $f : \mathbb{R}^{1+d} \to \mathbb{C}$ we will denote by $f_t : \mathbb{R}^d \to \mathbb{C}$ the function $f_t(\,\cdot\,) := f(t, \,\cdot\,)$.
- we denote by $\mathcal{S}(\mathbb{R}^{1+d})$ the Schwartz class in $\mathbb{R}^{1+d}$. If $f \in \mathcal{S}(\mathbb{R}^{1+d})$ we define its (unitary) Fourier transform:

$$\widehat{f}(E, p) := (2\pi)^{-(1+d)/2} \int \mathrm{e}^{\mathrm{i}(Et - p\cdot x)} f(t, x) dt dx, \tag{3.1}$$

so that

$$f(t, x) = (2\pi)^{-(1+d)/2} \int \mathrm{e}^{-\mathrm{i}(Et - p\cdot x)} \widehat{f}(E, p) dE dp. \tag{3.2}$$

Note the different sign in the exponent in comparison with (2.3).

If $f \in S(\mathbb{R}^d)$ we set:

$$\widehat{f}(p) = (2\pi)^{-d/2} \int \mathrm{e}^{-\mathrm{i}p\cdot x} f(x) dx,$$

and

$$\check{f}(x) = (2\pi)^{-d/2} \int \mathrm{e}^{\mathrm{i}p\cdot x} f(p) dp.$$

- If B is an observable, we write $B^{(*)}$ to denote either B or B^*. We will also set

$$B_t := B(t, 0),\ B(x) := B(0, x) \text{ so that } B(t, x) = B_t(x).$$

3.2. **Auxiliary maps $\mathbf{a}_B$.** For $B \in \mathfrak{A}$, $f \in \mathcal{S}(\mathbb{R}^d)$ we set:

$$B(f) := \int B(x) f(x) dx,$$

so that $B^*(f) = B(\overline{f})^*$. Clearly, if $B_1, B_2 \in \mathfrak{A}$ are almost local, then

$$\|[B_1(x_1), B_2(x_2)]\| \leq C_N \langle x_1 - x_2\rangle^{-N},\ \forall\ N \in \mathbb{N}. \tag{3.3}$$

This immediately implies that

$$\|[B_1(f_1), B_2(f_2)]\| \leq C_N \int |f_1(x_1)| \langle x_1 - x_2\rangle^{-N} |f_2(x_2)| dx_1 dx_2,\ f_1, f_2 \in \mathcal{S}(\mathbb{R}^d). \tag{3.4}$$

Now we introduce auxiliary maps which will be often used in our investigation:

Definition 3.1. *We denote by $a_B : \mathcal{H} \to \mathcal{S}'(\mathbb{R}^d; \mathcal{H})$ the linear operator defined as:*

$$a_B \Psi(x) := B(x)\Psi,\ x \in \mathbb{R}^d.$$

Clearly $a_B : \mathcal{H} \to \mathcal{S}'(\mathbb{R}^d; \mathcal{H})$ is continuous and

$$B(f) = (\mathbb{1}_{\mathcal{H}} \otimes \langle \overline{f}|) \circ a_B,\ f \in \mathcal{S}(\mathbb{R}^d), \tag{3.5}$$

where $(\mathbb{1}_{\mathcal{H}} \otimes \langle \overline{f}|) : \mathcal{S}'(\mathbb{R}^d; \mathcal{H}) \to \mathcal{H}$ is defined on simple tensors by

$$(\mathbb{1}_{\mathcal{H}} \otimes \langle \overline{f}|)(\Psi \otimes T) = T(f)\Psi,\ \Psi \in \mathcal{H}, T \in \mathcal{S}'(\mathbb{R}^d). \tag{3.6}$$

By duality $a_B^* : \mathcal{S}(\mathbb{R}^d; \mathcal{H}) \to \mathcal{H}$ is continuous and

$$B^*(f) = a_B^* \circ (\mathbb{1}_{\mathcal{H}} \otimes |f\rangle),\ f \in \mathcal{S}(\mathbb{R}^d). \tag{3.7}$$

The group of space translations

$$\tau_y \Psi(x) := \Psi(x - y),\ y \in \mathbb{R}^d,$$

is a strongly continuous group on $\mathcal{S}'(\mathbb{R}^d; \mathcal{H})$, and its generator is D_x i.e., $\tau_y = \mathrm{e}^{-\mathrm{i}y\cdot D_x}$. It is easy to check the following identity:

$$a_B \circ \mathrm{e}^{-\mathrm{i}y\cdot P} = \mathrm{e}^{-\mathrm{i}y\cdot(D_x + P)} \circ a_B,\ y \in \mathbb{R}^d. \tag{3.8}$$

We collect now some properties of a_B.

Lemma 3.2. *Let $B \in \mathfrak{A}$. Then:*

(1) *For any Borel set $\Delta \subset \mathbb{R}^{1+d}$:*

$$\begin{aligned} a_B \mathbb{1}_\Delta(U) = &\ (\mathbb{1}_{\overline{\Delta + \mathrm{supp}(\widehat{B})}}(U) \otimes \mathbb{1}_{\mathcal{S}'(\mathbb{R}^d)}) \circ a_B \mathbb{1}_\Delta(U), \\ a_B^* \circ (\mathbb{1}_\Delta(U) \otimes \mathbb{1}_{\mathcal{S}(\mathbb{R}^d)}) = &\ \mathbb{1}_{\overline{\Delta - \mathrm{supp}(\widehat{B})}}(U) a_B^* \circ (\mathbb{1}_\Delta(U) \otimes \mathbb{1}_{\mathcal{S}(\mathbb{R}^d)}). \end{aligned}$$

(2) *For any $f \in \mathcal{S}(\mathbb{R}^d)$ one has $f(D_x) a_B = a_{B_f}$ for*

$$\begin{aligned} B_f := &\ (2\pi)^{-d/2} \textstyle\int \check{f}(-y) B(0, y) dy = (2\pi)^{-(d+1)/2} \int f(-p) \widehat{B}(E, p) dE dp, \\ \widehat{B_f}(E, p) = &\ f(-p)\widehat{B}(E, p). \end{aligned}$$

(3) *If $\mathrm{supp}(\widehat{B})$ is compact and $f \in C^\infty(\mathbb{R}^d)$ then the above properties also hold.*

Proof. (1) follows from (2.4). (2) and (3) follow from the identity:

$$\mathrm{e}^{-\mathrm{i}y\cdot D_x}a_B = a_{B(0,-y)},\ y\in\mathbb{R}^d,$$

which is a rephrasing of (3.8). □

If $B\in\mathcal{L}_0$, then a_B has much stronger properties. In particular, for $\Delta \Subset \mathbb{R}^{1+d}$ the operator $a_B 1\!\!1_\Delta(U)$ maps $\mathcal{H}$ into $L^2(\mathbb{R}^d;\mathcal{H})\simeq \mathcal{H}\otimes L^2(\mathbb{R}^d)$, see Lemma 3.4 below. This is a consequence of the following important property of energy-decreasing operators, proven in [Bu90].

Lemma 3.3. *Let $B\in\mathfrak{A}$ be energy-decreasing with $\mathrm{supp}(\widehat{B})\Subset\mathbb{R}^{1+d}$ and $\Delta\subset\mathbb{R}^{1+d}$ be some bounded Borel set. Let $Y\subset\mathbb{R}^{1+d}$ be a subspace and let dy be the Lebesgue measure on Y. Then there exists $c>0$ such that for any $F\Subset Y$, one has:*

$$\| \int_F (B^*B)(y)1\!\!1_\Delta(U)dy\| \leq c\int_{F-F}\|[B^*,B(y)]\|dy. \tag{3.9}$$

Note that if B is in addition almost local and Y is spacelike, then the function $Y\ni y\mapsto \|[B^*,B(y)]\|$ vanishes faster than any inverse power of $|y|$ as $|y|\to\infty$, hence we can take $F=Y$ in (3.9). We will usually apply this lemma with $Y=\{0\}\times\mathbb{R}^d$. In view of this lemma, it is convenient to introduce the subspace of vectors with compact energy-momentum spectrum:

$$\mathcal{H}_{\mathrm{c}}(U) := \{\Psi\in\mathcal{H}\ :\ \Psi = 1\!\!1_\Delta(U)\Psi,\ \Delta\Subset\mathbb{R}^{1+d}\}.$$

We note the following simple fact:

Lemma 3.4. *Assume that $\Delta\Subset\mathbb{R}^{1+d}$ and let $B\in\mathcal{L}_0$. Then*

$$a_B 1\!\!1_\Delta(U):\mathcal{H}\to\mathcal{H}\otimes L^2(\mathbb{R}^d) \textit{ is bounded.}$$

Remark 3.5. *Considering a_B as a linear operator from $\mathcal{H}$ to $\mathcal{H}\otimes L^2(\mathbb{R}^d)$ with domain $\mathcal{H}_{\mathrm{c}}(U)$, we see that $\mathcal{H}\otimes\mathcal{S}(\mathbb{R}^d)\subset \mathrm{Dom}\ a_B^*$, hence a_B is closable.*

Proof. It suffices to note that

$$1\!\!1_\Delta(U)a_B^*\circ a_B 1\!\!1_\Delta(U) = \int_{\mathbb{R}^d} 1\!\!1_\Delta(U)(B^*B)(x)1\!\!1_\Delta(U)dx,$$

and use Lemma 3.3. □

3.3. **Particle detectors.** In this subsection we make contact with the particle detectors C_t introduced in (2.6).

Definition 3.6. *Let $B\in\mathcal{L}_0$. For $h\in B(L^2(\mathbb{R}^d))$ we set:*

$$N_B(h) := a_B^*\circ(1\!\!1_{\mathcal{H}}\otimes h)\circ a_B,\ \mathrm{Dom}\ N_B(h) = \mathcal{H}_{\mathrm{c}}(U).$$

Denoting by $h(x,y)$ the distributional kernel of h we have the following expression for $N_B(h)$,

$$N_B(h) = \int B^*(x)h(x,y)B(y)dxdy, \tag{3.10}$$

which makes sense as a quadratic form identity on $\mathcal{H}_{\mathrm{c}}(U)$. If h is the operator of multiplication by the function $x\mapsto h(x)$, then $N_B(h)$ can be written as

$$N_B(h) = \int (B^*B)(x)h(x)dx.$$

Setting $h_t(x) := h\left(\frac{x}{t}\right)$, we see that C_t defined in (2.6) equals $N_{B_t}(h_t)$, where $B_t = B(t,0)$. The following lemma is a direct consequence of Lemmas 3.2 and 3.4.

Lemma 3.7. *We have:*

(1) $\|N_B(h)1\!\!1_\Delta(U)\|_{B(\mathcal{H})}\leq c_{\Delta,B}\|h\|_{B(L^2(\mathbb{R}^d))}$,

(2) $\forall\ \Delta\Subset\mathbb{R}^{1+d},\ N_B(h)1\!\!1_\Delta(U) = 1\!\!1_{\Delta_1}(U)N_B(h)1\!\!1_\Delta(U),$ *for some* $\Delta_1\Subset\mathbb{R}^{1+d}$.

3.4. **Auxiliary maps $\mathbf{a}_{B_1,B_2}$.** We start with the following definition which is meaningful due to Lemma 3.4:

Definition 3.8. *If $B_1, B_2 \in \mathcal{L}_0$, then we can define the linear operator:*

$$a_{B_1,B_2}: \quad \begin{aligned} &\mathcal{H}_{\rm c}(U) \to \mathcal{H}\otimes L^2(\mathbb{R}^{2d}, dx_1dx_2),\\ &\Psi \mapsto a_{B_1,B_2}\Psi = (a_{B_1}\otimes \mathbb{1}_{L^2(\mathbb{R}^d)})\circ a_{B_2}\Psi. \end{aligned} \tag{3.11}$$

Formally we have

$$a_{B_1,B_2}\Psi(x_1,x_2) = B_1(x_1)B_2(x_2)\Psi.$$

We note the following lemma, which is a direct consequence of Lemmas 3.2 and 3.4.

Lemma 3.9. *Assume $\Delta \subset \mathbb{R}^{1+d}$ is compact and let $B_1, B_2 \in \mathcal{L}_0$. Then:*
(1) *$a_{B_1,B_2}\mathbb{1}_\Delta(U): \mathcal{H} \to \mathcal{H}\otimes L^2(\mathbb{R}^{2d}, dx_1dx_2)$ is bounded,*
(2) *for any $\Delta \Subset \mathbb{R}^{1+d}$ one has:*

$$a_{B_1,B_2}\mathbb{1}_\Delta(U) = (\mathbb{1}_{\Delta+\mathrm{supp}(\widehat{B}_1)+\mathrm{supp}(\widehat{B}_2)}(U)\otimes \mathbb{1}_{L^2(\mathbb{R}^{2d})})\circ a_{B_1,B_2}\mathbb{1}_\Delta(U),$$
$$a^*_{B_1,B_2}\circ(\mathbb{1}_\Delta(U)\otimes \mathbb{1}_{L^2(\mathbb{R}^{2d})}) = \mathbb{1}_{\Delta-\mathrm{supp}(\widehat{B}_1)-\mathrm{supp}(\widehat{B}_2)}(U)a^*_{B_1,B_2}\circ(\mathbb{1}_\Delta(U)\otimes \mathbb{1}_{L^2(\mathbb{R}^{2d})}).$$

For later use we state in Lemma 3.10 below a simple consequence of almost locality. To simplify the formulation of this result, we introduce the following functions for $N > d$:

$$g_N(k) = \int \mathrm{e}^{-\mathrm{i}k\cdot x}\langle x\rangle^{-N}dx. \tag{3.12}$$

Clearly

$$\partial_k^\alpha g_N(k) \in O(\langle k\rangle^{-p}),\ \forall\, p\in\mathbb{N},\ |\alpha| < N-|d|,$$

and the operator on $L^2(\mathbb{R}^d)$ with kernel $\langle x-y\rangle^{-N}$ equals $g_N(D_x)$.

Lemma 3.10. *Let $\Delta \Subset \mathbb{R}^{1+d}$, $B_i \in \mathcal{L}_0$, $h_i \in C_0^\infty(\mathbb{R}^d)$, $i = 1, 2$. We denote by $h_i \in B(L^2(\mathbb{R}^d))$ the operator of multiplication by h_i. Then for any $N \in \mathbb{N}$ one has:*

$$\|\big(N_{B_1}(h_1)\circ N_{B_2}(h_2) - a^*_{B_2,B_1}\circ(\mathbb{1}_\mathcal{H}\otimes h_1\otimes h_2)\circ a_{B_1,B_2}\big)\mathbb{1}_\Delta(U)\| \le C_{N,\Delta,B_1,B_2}\|h_1 g_N(D_x)h_2\|_{B(L^2(\mathbb{R}^d))}. \tag{3.13}$$

Remark 3.11. *In applications we will often estimate the operator norm on the r.h.s. of (3.13) by the Hilbert-Schmidt norm $\|\cdot\|_{\rm HS}$.*

Proof. Let R be the operator in the l.h.s. of (3.13). By Lemmas 3.7, 3.9 $R = \mathbb{1}_{\Delta_1}(U)R\mathbb{1}_{\Delta_2}(U)$ for some $\Delta_i \Subset \mathbb{R}^{1+d}$. For $u_i \in \mathcal{H}$ we have

$$\begin{aligned} &|(u_1|Ru_2)_\mathcal{H}|\\ =\ & |\textstyle\int(\mathbb{1}_{\Delta_1}(U)u_1|B_1^*(x_1))[B_1(x_1), B_2^*(x_2)]B_2(x_2)h_1(x_1)h_2(x_2)\mathbb{1}_{\Delta_2}(U)u_2)_\mathcal{H}dx_1dx_2|\\ \le\ & C\textstyle\int\|B_1(x_1)\mathbb{1}_{\Delta_1}(U)u_1\|_\mathcal{H}\|B_2(x_2)\mathbb{1}_{\Delta_2}(U)u_2\|_\mathcal{H}|h_1|(x_1)|h_2|(x_2)\langle x_1-x_2\rangle^{-N}dx_1dx_2. \end{aligned}$$

By Lemma 3.7 we know that $v_i(x) = \|B_i(x)\mathbb{1}_{\Delta_i}(U)u_i\|_\mathcal{H} \in L^2(\mathbb{R}^d)$ with $\|v_i\|_{L^2(\mathbb{R}^d)} \le C_i\|u_i\|_\mathcal{H}$. Therefore

$$|(u_1|Ru_2)|_\mathcal{H} \le C\||h_1|g_N(D_x)|h_2|\|_{B(L^2(\mathbb{R}^d))}\|u_1\|_\mathcal{H}\|u_2\|_\mathcal{H}.$$

Writing $h_i = |h_i|\mathrm{sign}(h_i)$ and using that the operator of multiplication by $\mathrm{sgn}(h_i)$ is unitary, we obtain the lemma. □

4. An intermediate convergence argument

For $B \in \mathcal{L}_0$ and $h \in C_0^\infty(\mathbb{R}^d)$ we set:

$$h_t(x) := h\left(\frac{x}{t}\right),\ N_B(h,t) := N_{B_t}(h_t). \tag{4.1}$$

Recalling the notation $\tilde{x} = (x_1, x_2)$, we also define $\tilde{\omega}(D_{\tilde{x}}) = \omega(D_{x_1}) + \omega(D_{x_2})$, acting on $L^2(\mathbb{R}^{2d})$. The following theorem is an important step in the proofs of Thms. 2.6 and 2.7. It essentially allows to reduce their proofs to arguments adapted from non-relativistic scattering theory.

Theorem 4.1. *Let $\Delta \subset \mathbb{R}^{1+d}$ be a bounded open set, $B_1, B_2 \in \mathcal{L}_0$ with (B_1, B_2) $\Delta-$admissible and let $h_1, h_2 \in C_0^\infty(\mathbb{R}^d)$ have disjoint supports. Let*

$$\mathrm{H}_t(x_1, x_2) := h_{1,t}(x_1) h_{2,t}(x_2) \tag{4.2}$$

and set for $\Psi \in 1\!\!1_\Delta(U)\mathcal{H}$:

$$F_t := (\langle \Omega | \otimes 1\!\!1_{L^2(\mathbb{R}^{2d})}) \circ a_{B_1,B_2} \mathrm{e}^{-\mathrm{i}tH} \Psi \in L^2(\mathbb{R}^{2d}), \tag{4.3}$$

so that

$$F_t(x_1, x_2) = (\Omega | B_1(t, x_1) B_2(t, x_2) \Psi)_{\mathcal{H}}, \ (x_1, x_2) \in \mathbb{R}^{2d}.$$

Assume that:

$$F_+ := \lim_{t \to \infty} \mathrm{e}^{\mathrm{i}t\tilde{\omega}(D_{\tilde{x}})} \mathrm{H}_t F_t \ \textit{exists.} \tag{4.4}$$

Then

$$\lim_{t \to \infty} N_{B_1}(h_1, t) N_{B_2}(h_2, t) \Psi \tag{4.5}$$

exists and belongs to $1\!\!1_\Delta(U)\mathcal{H}_2^+$.

Proof. Applying Lemma 3.10 and noting that $\|h_{1,t} g_N(D_x) h_{2,t}\|_{\mathrm{HS}} \in O(t^{d-N})$, we get:

$$N_{B_1}(h_{1,t}) N_{B_2}(h_{2,t}) 1\!\!1_\Delta(U) = a^*_{B_2,B_1} \circ (1\!\!1_{\mathcal{H}} \otimes \mathrm{H}_t) \circ a_{B_1,B_2} 1\!\!1_\Delta(U) + O(t^{-\infty}).$$

By (2.10) and Lemma 3.9 we have:

$$a_{B_1,B_2} 1\!\!1_\Delta(U) = a(1\!\!1_{\{0\}}(U) \otimes 1\!\!1_{L^2(\mathbb{R}^{2d})}) \circ a_{B_1,B_2} 1\!\!1_\Delta(U) = (|\Omega\rangle\langle\Omega| \otimes 1\!\!1_{L^2(\mathbb{R}^{2d})}) \circ a_{B_1,B_2} 1\!\!1_\Delta(U),$$

using (2.2). Therefore we have:

$$\begin{aligned} & \mathrm{e}^{\mathrm{i}tH} N_{B_1}(h_{1,t}) N_{B_2}(h_{2,t}) \mathrm{e}^{-\mathrm{i}tH} \Psi = \mathrm{e}^{\mathrm{i}tH} a^*_{B_2,B_1}(\Omega \otimes \mathrm{H}_t F_t) + O(t^{-\infty}) \\ = \ & \mathrm{e}^{\mathrm{i}tH} a^*_{B_2,B_1}(\Omega \otimes \mathrm{e}^{-\mathrm{i}t\tilde{\omega}(D_{\tilde{x}})} F_+) + o(t^0). \end{aligned} \tag{4.6}$$

Set

$$S_t : L^2(\mathbb{R}^{2d}) \ni F \mapsto \ \mathrm{e}^{\mathrm{i}tH} a^*_{B_2,B_1}(\Omega \otimes \mathrm{e}^{-\mathrm{i}t\tilde{\omega}(D_{\tilde{x}})} F) \in \mathcal{H}.$$

By Lemma 3.9 the family S_t is uniformly bounded in norm. Moreover if g_1, g_2 are two positive energy KG solutions with disjoint velocity supports (see Subsect. 6.1 for terminology) and $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$ are their initial data, then

$$S_t(f_1 \otimes f_2) = B^*_{1,t}(g_1) B^*_{2,t}(g_2) \Omega,$$

where the Haag-Ruelle creation operators $B^*_{i,t}(g_i)$ are defined in Subsect. 6.2. From Thm. 6.5 we know that $\lim_{t\to\infty} S_t(f_1 \otimes f_2)$ exists. By linearity and density, using the uniform boundedness of S_t, we conclude that $\lim_{t\to\infty} S_t F$ exists for any $F \in L^2(\mathbb{R}^{2d})$. By (4.6) this implies the existence of the limit in (4.5). The approximation argument above implies that this limit belongs to $\mathcal{H}_2^+$. The fact that it belongs to the range of $1\!\!1_\Delta(U)$ follows from the $\Delta-$admissibility of (B_1, B_2).□

The proof of the existence of the limit (4.4) will be given in the next section. As a preparation, we collect some properties of the vectors $F_t \in L^2(\mathbb{R}^{2d})$. The most important property is that F_t solves a Schrödinger equation with Hamiltonian $\tilde{\omega}(D_{\tilde{x}})$ and a *source term* $\langle R \rangle_t$ whose L^2 norm outside of the diagonal decreases very fast when $t \to +\infty$.

Lemma 4.2. *Let F_t be defined in (4.3). Then:*
(1) *F_t is uniformly bounded in $L^2(\mathbb{R}^{2d})$,*
(2) *$t \mapsto F_t \in L^2(\mathbb{R}^{2d})$ is C^1 with*

$$\partial_t F_t = -\mathrm{i}\tilde{\omega}(D_{\tilde{x}}) F_t + \langle R \rangle_t,$$

where $\|\tilde{\mathrm{H}}\left(\frac{\tilde{x}}{t}\right) \langle R \rangle_t\|_{L^2(\mathbb{R}^{2d})} \in O(t^{-\infty})$ for any $\tilde{\mathrm{H}} \in C_0^\infty(\mathbb{R}^{2d})$ with $\mathrm{supp}\tilde{\mathrm{H}} \cap \{x_1 = x_2\} = \emptyset$.

Proof. We have $F_t(x_1,x_2) = (\Omega|B_1(t,x_1)B_2(t,x_2)\Psi)_{\mathcal{H}}$ and from Lemma 3.9 we know that F_t is uniformly bounded in $L^2(\mathbb{R}^{2d})$. Moreover, since $\Psi \in \mathcal{H}_{\rm c}(U)$, we see that $t \mapsto F_t \in L^2(\mathbb{R}^{2d})$ is C^1 with:

$$\begin{aligned}\partial_t F_t = &\ (\Omega|\dot{B}_1(t,x_1)B_2(t,x_2)\Psi)_{\mathcal{H}} + (\Omega|B_1(t,x_1)\dot{B}_2(t,x_2)\Psi)_{\mathcal{H}} \\ = &\ (\Omega|\dot{B}_1(t,x_1)B_2(t,x_2)\Psi)_{\mathcal{H}} + (\Omega|\dot{B}_2(t,x_2)B_1(t,x_1)\Psi)_{\mathcal{H}} \\ &+(\Omega|[B_1(t,x_1),\dot{B}_2(t,x_2)]\Psi)_{\mathcal{H}},\end{aligned}$$

where $\dot{B}_i := \partial_s B_i(s,0)_{|s=0}$ are again almost local by the definition of $\mathcal{L}_0$. We have for any $\Phi \in \mathcal{H}$:

$$\begin{aligned}(\Omega|B_j(t,x_j)\Phi)_{\mathcal{H}} = &\ (\Omega|1\!\!1_{\{0\}}(U)B_j(t,x_j)\Phi)_{\mathcal{H}} = (\Omega|B_j(t,x_j)1\!\!1_{H_m}(U)\Phi)_{\mathcal{H}} \\ = &\ (\Omega|B_j(x_j)\mathrm{e}^{-\mathrm{i}t\omega(P)}\Phi)_{\mathcal{H}} = \mathrm{e}^{-\mathrm{i}t\omega(D_{x_j})}(\Omega|B_j(x_j)\Phi),\end{aligned}$$

using (2.4), (2.8) and finally (3.8). Differentiating this identity we obtain

$$(\Omega|\dot{B}_j(t,x_j)\Phi)_{\mathcal{H}} = -\mathrm{i}\omega(D_{x_j})(\Omega|B_j(t,x_j)\Phi)_{\mathcal{H}}.$$

Therefore we get:

$$\begin{aligned}\partial_t F_t = &\ -\mathrm{i}\omega(D_{x_1})(\Omega|B_1(t,x_1)B_2(t,x_2)\Psi)_{\mathcal{H}} - \mathrm{i}\omega(D_{x_2})(\Omega|B_2(t,x_2)B_1(t,x_1)\Psi)_{\mathcal{H}} \\ &+(\Omega|[B_1(t,x_1),\dot{B}_2(t,x_2)]\Psi)_{\mathcal{H}} \\ = &\ -\mathrm{i}\omega(D_{x_1})(\Omega|B_1(t,x_1)B_2(t,x_2)\Psi)_{\mathcal{H}} - \mathrm{i}\omega(D_{x_2})(\Omega|B_1(t,x_1)B_2(t,x_2)\Psi)_{\mathcal{H}} \\ &-\mathrm{i}\omega(D_{x_2})(\Omega|[B_2(t,x_2),B_1(t,x_1)]\Psi)_{\mathcal{H}} + (\Omega|[B_1(t,x_1),\dot{B}_2(t,x_2)]\Psi)_{\mathcal{H}} \\ = &\ -\mathrm{i}\tilde{\omega}(D_{\tilde{x}})F_t + \langle R\rangle_t,\end{aligned}$$

for

$$\begin{aligned}\langle R\rangle_t = &\ -\mathrm{i}\omega(D_{x_2})(\Omega|[B_2(t,x_2),B_1(t,x_1)]\Psi)_{\mathcal{H}} + (\Omega|[B_1(t,x_1),\dot{B}_2(t,x_2)]\Psi)_{\mathcal{H}} \\ =: &\ \langle R\rangle_{1,t} + \langle R\rangle_{2,t}.\end{aligned}$$

Since $\dot{B}_2$ is almost local, we have $\|[B_1(t,x_1),\dot{B}_2(t,x_2)]\| \in O(\langle x_1-x_2\rangle^{-N})$ uniformly in t and $\|\tilde{\mathrm{H}}_t\langle R\rangle_{2,t}\|_{L^2(\mathbb{R}^{2d})} \in O(t^{-\infty})$ because of the support properties of $\tilde{\mathrm{H}}_t(\tilde{x}) := \tilde{\mathrm{H}}\left(\frac{\tilde{x}}{t}\right)$.

To estimate $\langle R\rangle_{1,t}$ we write it as $(\Omega|[\omega(D_{x_2})B_2(t,x_2),B_1(t,x_1)]\Psi)_{\mathcal{H}}$. By Lemma 3.2 (2) we see that $\omega(D_{x_2})B_2(t,x_2) = C_2(t,x_2)$, where

$$C_2 = (2\pi)^{-d/2}\int f(x)B(0,x)dx,\ \ f \in \mathcal{S}(\mathbb{R}^d),\ \ \widehat{f}(-p) \equiv \omega(p) \text{ near } \mathrm{supp}(\widehat{B}_2).$$

Therefore C_2 is almost local and $\|[C_2(t,x_2),B_1(t,x_1)]\| \in O(\langle x_1-x_2\rangle^{-N})$. The same argument as above shows that $\|\tilde{\mathrm{H}}_t\langle R\rangle_{1,t}\|_{L^2(\mathbb{R}^{2d})} \in O(t^{-\infty})$. □

5. Non-relativistic scattering with source terms

In this section we give the proof of the existence of the limit

$$F_+ = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\tilde{\omega}(D_{\tilde{x}})}\mathrm{H}_t F_t,$$

appearing in Thm. 4.1. The proof is obtained by adapting to our situation the standard arguments based on *propagation estimates*. The main difference with the usual scattering theory is that F_t solves a Schrödinger equation with a *source term*. This implies that one has to use propagation observables supported in regions where the source term is small, in our case outside the diagonal in $\mathbb{R}^{2d}$. The necessary abstract arguments are collected in Appendix A.

5.1. **Large velocity estimates.** In this subsection we prove large velocity estimates. Note that we do not prove them directly for F_t, but use instead a general argument based on Lemma 3.3, locality and the fact that the hyperplanes $\{t = v \cdot x\}$ for $|v| > 1$ are space-like.

Lemma 5.1. *Let $B \in \mathcal{L}_0$, $\Delta \Subset \mathbb{R}^{1+d}$ and $1 < c < C$. Then,*

$$\int_1^{+\infty} (\mathrm{e}^{-\mathrm{i}tH}\Psi | \mathbb{1}_\Delta(U) N_B(\mathbb{1}_{\{z\in\mathbb{R}^d \;:\; c\le|z|\le C\}}\left(\frac{x}{t}\right))\mathbb{1}_\Delta(U)\mathrm{e}^{-\mathrm{i}tH}\Psi)_{\mathcal{H}}\frac{dt}{t} \le c_1\|\Psi\|^2_{\mathcal{H}},\ \Psi\in\mathcal{H},$$

where x in the formula above denotes the corresponding multiplication operator on $L^2(\mathbb{R}^d)$.

Proof. Set $z = (z^1, z') \in \mathbb{R}^d$ where $z^1 \in \mathbb{R}$ is the first component of z. We can find constants $c_i > 1$ and rotations $R_i \in SO(\mathbb{R}^d)$ such that

$$\{z \;:\; c \le |z| \le C\} \subset \textstyle\bigcup_{i=1}^N \{z \;:\; c_i \le |(R_i z)^1| \le C\}.$$

So it suffices to prove the lemma with $\mathbb{1}_{\{z \,:\, c\le|z|\le C\}}$ replaced with $\mathbb{1}_{\{z \,:\, c\le|(Rz)^1|\le C\}}$ for $c > 1$, $R \in SO(\mathbb{R}^d)$. We parametrize the set $S = \{z \;:\; c_i \le |(Rz)^1| \le C\}$ by coordinates (y^1, y') with $y^1 = (Rx)^1$ so that it equals $S = \{(y^1, y') \;:\; c \le |y^1| \le C\}$. We have:

$$\begin{aligned} I :=&\ \textstyle\int_1^\infty \mathrm{e}^{\mathrm{i}tH} N_B(\mathbb{1}_S\left(\frac{x}{t}\right))\mathrm{e}^{-\mathrm{i}tH}\frac{dt}{t} = \int_1^\infty \frac{dt}{t}\int_{\mathbb{R}^d} \mathbb{1}_S(\frac{y}{t})(B^*B)(t,y)dy \\ =&\ \textstyle\int_1^\infty dt \int_c^C dv \int_{\mathbb{R}^{d-1}} (B^*B)(t,tv,y')dy' = \int_c^C dv \int_{\mathbb{R}^d}(B^*B)(t,tv,y')dtdy'. \end{aligned}$$

We now apply Lemma 3.3 to the subspace $Y_v = \{(t, tv, y') \;:\; t\in\mathbb{R},\ y'\in\mathbb{R}^{d-1}\}$ for $c \le v \le C$ which yields:

$$\|\mathbb{1}_\Delta(U) I \mathbb{1}_\Delta(U)\| \le C' \int_c^C dv \int_{\mathbb{R}^d} \|[B^*, B(t,tv,y')]\| dtdy'. \tag{5.1}$$

Since B is almost local, there exist $B_r \in \mathfrak{A}(\mathcal{O}_r)$ with $\|B - B_r\| \in O(\langle r\rangle^{-n})$. Therefore

$$\|[B^*, B(t,tv,y')]\| \le C\langle r\rangle^{-n} + \|[B_r^*, B_r(t,tv,y')]\|.$$

Set $u \cdot u = x^2 - t^2$ for $u = (t,x) \in \mathbb{R}^{1+d}$. If $v_1, v_2 \in \mathcal{O}_r$ and $u_1 = v_1 + (t,tv,y')$, $u_2 = v_2$, then $u = u_1 - u_2 = (t,tv,y') + w$, for $w \in \mathcal{O}_r - \mathcal{O}_r \subset \mathcal{O}_{2r}$. It follows that

$$u\cdot u = t^2(|v|^2-1) + |y'|^2 + O(r)(\langle t\rangle + \langle y'\rangle) + O(r^2).$$

Using that $c > 1$, we conclude that there exists $0 < \delta \ll 1$ such that if $\langle r\rangle = \delta(\langle t\rangle + \langle y'\rangle)$ then $\mathcal{O}_r$ and $\mathcal{O}_r + (t,tv,y')$ are spacelike separated for any $(t,y') \in \mathbb{R}^d$ s.t. $t^2 + |y'|^2 \ge 1$ and $c \le v \le C$. Therefore $\|[B^*, B(t,tv,y')]\| \in O(\langle t\rangle + \langle y'\rangle)^{-n}$, and the integral in the r.h.s. of (5.1) is finite. □

To proceed we need the following definitions: For $0 \le r_1 < r_2$ and $\epsilon \ge 0$ we set:

$$C_{r_1,r_2} := \{\tilde{x} \in \mathbb{R}^{2d} \;:\; r_1 \le |\tilde{x}| \le r_2\},\ C_r := C_{0,r},\ D_\epsilon := \{\tilde{x}\in\mathbb{R}^{2d} \;:\; |x_1 - x_2| \le \epsilon\}.$$

Let us now prove the following corollary of Lemma 5.1:

Proposition 5.2. *Let $\sqrt{2} < r < r'$, $\epsilon > 0$ and let F_t be defined in (4.3). Then*

$$\int_1^{+\infty} \left\| \mathbb{1}_{C_{r,r'}\backslash D_\epsilon}\left(\frac{\tilde{x}}{t}\right) F_t \right\|^2_{L^2(\mathbb{R}^{2d})} \frac{dt}{t} < \infty,$$

where $\tilde{x}$ in the formula above denotes the corresponding multiplication operator on $L^2(\mathbb{R}^{2d})$.

Proof. Set $\tilde{x} = (x_1, x_2) \in \mathbb{R}^{2d}$. By a covering argument, it suffices to prove the lemma with $\mathbb{1}_{C_{r,r'}\backslash D_\epsilon}(\tilde{x})$ replaced with $h_1(x_1)h_2(x_2)$, where $h_i \in C_0^\infty(\mathbb{R}^d)$ are supported near some points $y_i \in \mathbb{R}^d$ with $(y_1, y_2) \in C_{r,r'}\backslash D_\epsilon$ and $\mathrm{d}(\mathrm{supp}h_1, \mathrm{supp}h_2) > 0$. Set $\mathrm{H}_t(\tilde{x}) = h_1(\frac{x_1}{t})h_2(\frac{x_2}{t})$. By (4.3) we have:

$$(F_t|\mathrm{H}_t F_t)_{L^2(\mathbb{R}^{2d})} = \int_{\mathbb{R}^{2d}} (\mathrm{e}^{-\mathrm{i}tH}\Psi | B_2^*(x_2)B_1^*(x_1)B_1(x_1)B_2(x_2)\mathrm{e}^{-\mathrm{i}tH}\Psi)_{\mathcal{H}} h_1\left(\frac{x_1}{t}\right) h_2\left(\frac{x_2}{t}\right) dx_1dx_2.$$

Since $|(y_1,y_2)|>\sqrt{2}$, necessarily $|y_i|>1$ either for $i=1$ or $i=2$, and we can assume that $\mathrm{supp} h_i\subset\{y\in\mathbb{R}^d\ :\ |y|>1\}$. If this holds for $i=2$ then

$$\begin{aligned}(F_t|\mathrm{H}_tF_t)_{L^2(\mathbb{R}^{2d})}\leq&\quad C\int(\mathrm{e}^{-\mathrm{i}tH}\Psi|B_2^*(x_2)B_2(x_2)\mathrm{e}^{-\mathrm{i}tH}\Psi)_{\mathcal{H}}h_2(\tfrac{x_2}{t})dx_2\\ \leq&\quad C(\mathrm{e}^{-\mathrm{i}tH}\Psi|N_{B_2}(h_2(\tfrac{x}{t}))\mathrm{e}^{-\mathrm{i}tH}\Psi)_{\mathcal{H}},\end{aligned}$$

where x denotes the corresponding multiplication operator on $\mathbb{R}^d$. Then we apply Lemma 5.1. If the above property holds for $i=1$ then using almost locality as in the proof of Lemma 3.10 we obtain that

$$\begin{aligned}&(F_t|\mathrm{H}_tF_t)_{L^2(\mathbb{R}^{2d})}\\ =&\quad\int_{\mathbb{R}^{2d}}(\mathrm{e}^{-\mathrm{i}tH}\Psi|B_1^*(x_1)B_2^*(x_2)B_2(x_2)B_1(x_1)\mathrm{e}^{-\mathrm{i}tH}\Psi)_{\mathcal{H}}h_1(\tfrac{x_1}{t})h_2(\tfrac{x_2}{t})dx_1dx_2+O(t^{-\infty})\\ =&\quad\int_{\mathbb{R}^{d}}(B_1(x_1)\mathrm{e}^{-\mathrm{i}tH}\Psi|N_{B_2}(h_2(\tfrac{x}{t}))B_1(x_1)\mathrm{e}^{-\mathrm{i}tH}\Psi)_{\mathcal{H}}h_1(\tfrac{x_1}{t})dx_1+O(t^{-\infty})\\ \leq&\quad C(\mathrm{e}^{-\mathrm{i}tH}\Psi|N_{B_1}(h_1(\tfrac{x}{t}))\mathrm{e}^{-\mathrm{i}tH}\Psi)_{\mathcal{H}}+O(t^{-\infty}),\end{aligned}$$

using that h_1, h_2 have disjoint supports. We complete the proof as before. □

5.2. **Phase-space propagation estimates.** We start with a geometrical consideration related to a well-known construction of Graf [Gr90].

Lemma 5.3. *Let $K\Subset\mathbb{R}^{2d}\backslash D_0$. Then there exist $\sqrt{2}<r<r'$, $c_1,c_2,\epsilon>0$ and a function $R\in C_0^\infty(\mathbb{R}^{2d})$ vanishing near D_0 such that*

$$\nabla^2R(\tilde{x})\geq c_1\mathbb{1}_K(\tilde{x})-c_2\mathbb{1}_{C_{r,r'}\backslash D_\epsilon}(\tilde{x}).\tag{5.2}$$

Proof. Set $\tilde{x}=(x_1,x_2)\in\mathbb{R}^{2d}$, $u=\frac{1}{\sqrt{2}}(x_1+x_2)$, $v=\frac{1}{\sqrt{2}}(x_1-x_2)$. We choose $\sqrt{2}<r<r'$ such that $K\subset C_r$ and set

$$g(\tilde{x})=(u^2+\beta v^2-c)F(\tilde{x}),$$

for $F\geq0$, $F\in C_0^\infty(C_{r_1'})$, $F\equiv1$ in C_{r_1} where $r<r_1<r_1'<r'$. The constants $c,\beta>0$ will be determined later. Note that g is convex in C_{r_1}, hence

$$R_0(\tilde{x})=\sup\{g,0\}(\tilde{x})$$

is convex in C_{r_1} (but not smooth). We first fix $c=r'^2$ so that $R_0(\tilde{x})=0$ for $\tilde{x}\in D_{\epsilon_\beta}$, for some $\epsilon_\beta>0$ tending to 0 when $\beta\to+\infty$. We choose then $\beta\gg1$ such that $K\subset\{\tilde{x}\in\mathbb{R}^{2d}\ :\ R_0(\tilde{x})>0\}$ and set $\epsilon=\epsilon_\beta$. By the continuity of R_0 we also obtain:

$$K\subset\bigcap_{|\tilde{x}'|\leq\epsilon'}\{\tilde{x}\ :\ R_0(\tilde{x}-\tilde{x}')>0\},\tag{5.3}$$

$$D_{\epsilon/2}\subset\bigcap_{|\tilde{x}'|\leq\epsilon'}\{\tilde{x}\ :\ R_0(\tilde{x}-\tilde{x}')=0\},\tag{5.4}$$

for some $\epsilon'\ll1$.

We now choose $\eta\geq0$, $\eta\in C_0^\infty(C_{\epsilon'})$ with $\int\eta(\tilde{x})d\tilde{x}=1$ and set:

$$R(\tilde{x}):=\int\eta(\tilde{x}')R_0(\tilde{x}-\tilde{x}')d\tilde{x}'=\eta\star R_0(\tilde{x}).$$

Clearly $R\in C_0^\infty(\mathbb{R}^{2d})$ and R is convex in C_r, hence

$$\nabla^2R(\tilde{x})\geq0,\ \tilde{x}\in C_r.\tag{5.5}$$

By relation (5.3), $R=\eta\star g$ on K, hence

$$\nabla^2R(\tilde{x})\geq c_1\mathbb{1},\ \tilde{x}\in K,\tag{5.6}$$

for some $c_1>0$. In $C_{r,r'}$, ∇^2R is bounded, and outside of $C_{r'}$, $\nabla^2R(\tilde{x})\geq0$ since $R(\tilde{x})\equiv0$ there by construction. By (5.5), (5.6) we obtain (5.2). □

Proposition 5.4. *Let F_t be defined in (4.3) and $K \Subset \mathbb{R}^{2d}\backslash D_0$. Then*

$$\int_1^{+\infty} \left\| 1\!\!1_K\left(\frac{\tilde{x}}{t}\right)\left(\frac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)F_t\right\|^2_{L^2(\mathbb{R}^{2d})}\frac{dt}{t} < \infty.$$

Proof. We will apply Lemma A.1 to $\mathcal{H} = L^2(\mathbb{R}^{2d})$, $u(t) = F_t$, $H = \tilde{\omega}(D_{\tilde{x}})$ and

$$M(t) = R\left(\frac{\tilde{x}}{t}\right) - \frac{1}{2}\left(\nabla R\left(\frac{\tilde{x}}{t}\right)\cdot\left(\frac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right) + \text{h.c.}\right).$$

Recall that $\mathcal{D}M(t)$ denotes the associated Heisenberg derivative. By standard pseudo-differential calculus we obtain that:

$$\begin{aligned} \mathcal{D}M(t) = \ & \tfrac{1}{t}\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)\cdot\nabla^2 R\left(\tfrac{\tilde{x}}{t}\right)\cdot\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right) + O(t^{-2}) \\ \geq \ & \tfrac{c_1}{t}\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)1\!\!1_K\left(\tfrac{\tilde{x}}{t}\right)\cdot\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right) - \tfrac{C}{t}1\!\!1_{C_{r,r'}}\left(\tfrac{\tilde{x}}{t}\right) + O(t^{-2}), \end{aligned} \tag{5.7}$$

where $O(t^{-2})$ denotes a term with norm $O(t^{-2})$ and we have used Lemma 5.3 in the second line. Since R is supported away from the diagonal, we obtain by Lemma 4.2 and pseudo-differential calculus that $\|M(t)\langle R\rangle_t\| \in L^1(\mathbb{R}^+, dt)$, where we recall that $\partial_t F_t =: -\mathrm{i}\tilde{\omega}(D_{\tilde{x}})F_t + \langle R\rangle_t$. Lemma 4.2 also gives that $\sup_t \|F_t\| < \infty$. The negative term in the r.h.s. of (5.7) is controlled by Proposition 5.2. Applying Lemma A.1 we obtain the desired result.□

5.3. **Existence of the intermediate limit.**

Theorem 5.5. *Let F_t, H_t be defined in (4.3). Then the limit*

$$F_+ = \lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\tilde{\omega}(D_{\tilde{x}})}\mathrm{H}_t F_t \textit{ exists.}$$

Proof. All the norms and scalar products in this proof are in the sense of $L^2(\mathbb{R}^{2d})$. We proceed as in the proof of [DG97, Prop. 4.4.5]. Set first $\mathrm{H}(\tilde{x}) = h_1(x_1)h_2(x_2)$ and

$$M(t) = \mathrm{H}\left(\frac{\tilde{x}}{t}\right) - \left(\frac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)\cdot\nabla\mathrm{H}\left(\frac{\tilde{x}}{t}\right).$$

By pseudo-differential calculus, we obtain that

$$\begin{aligned} &\mathcal{D}M(t) = \tfrac{1}{t}\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)\cdot\nabla^2\mathrm{H}\left(\tfrac{\tilde{x}}{t}\right)\cdot\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right) + O(t^{-2}), \\ &\|M(t)\langle R\rangle_t\|, \|M^*(t)\langle R\rangle_t\| \in L^1(\mathbb{R}^+, dt), \end{aligned} \tag{5.8}$$

where in the second line we use that H is supported away from the diagonal. Note that the following analog of Prop. 5.4 is well-known and easy to prove by mimicking the arguments in [DG97, Prop. 4.4.3]:

$$\int_1^{+\infty}\left\|1\!\!1_K\left(\frac{\tilde{x}}{t}\right)\left(\frac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)\mathrm{e}^{-\mathrm{i}t\tilde{\omega}(D_{\tilde{x}})}u\right\|^2\frac{dt}{t} \leq C\|u\|^2,\ u \in L^2(\mathbb{R}^{2d}), \tag{5.9}$$

for any $K \Subset \mathbb{R}^{2d}\backslash\{0\}$. Combining this estimate with the one in Prop. 5.4, we obtain by Lemma A.3 that

$$\lim_{t\to+\infty}\mathrm{e}^{\mathrm{i}t\tilde{\omega}(D_{\tilde{x}})}M(t)F_t \text{ exists.}$$

Therefore the proposition follows if we show that

$$\lim_{t\to\infty}\left(\frac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)\cdot\nabla\mathrm{H}\left(\frac{\tilde{x}}{t}\right)F_t = 0,$$

or equivalently

$$\lim_{t\to+\infty}(F_t|\left(\frac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)\tilde{G}\left(\frac{\tilde{x}}{t}\right)\left(\frac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)F_t)_{L^2(\mathbb{R}^{2d})} = 0, \tag{5.10}$$

for $\tilde{G} = \tilde{H}1\!\!1$, $\tilde{H} \in C_0^\infty(\mathbb{R}^{2d}\backslash D_0)$ and $\tilde{H} \geq 0$. It suffices to prove that the limit in (5.10) exists, since it will then be equal to 0 by Prop. 5.4. To this end, we apply Lemma A.2 with

$$M(t) = \left(\frac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)\tilde{G}\left(\frac{\tilde{x}}{t}\right)\left(\frac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right).$$

Again $\|M(t)\langle R\rangle_t\|, \|M^*(t)\langle R\rangle_t\| \in L^1(\mathbb{R}^+, dt)$ and by pseudo-differential calculus we have:

$$\begin{aligned}\mathcal{D}M(t) = & -\tfrac{2}{t}\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)\tilde{G}\left(\tfrac{\tilde{x}}{t}\right)\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)\\ & -\tfrac{1}{t}\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)\nabla\tilde{G}\left(\tfrac{\tilde{x}}{t}\right)\cdot\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right) + O(t^{-2})\\ = & \ \tfrac{1}{t}\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right)\mathbb{1}_K\left(\tfrac{\tilde{x}}{t}\right)A(t)\mathbb{1}_K\left(\tfrac{\tilde{x}}{t}\right)\left(\tfrac{\tilde{x}}{t} - \nabla\tilde{\omega}(D_{\tilde{x}})\right) + O(t^{-2}),\end{aligned}$$

for a compact set $K \subset \mathbb{R}^{2d}\backslash D_0$ and $A(t) \in O(1)$. Now the existence of the limit follows from Prop. 5.4 and Lemma A.2. □

6. Haag-Ruelle scattering theory

In this section we recall some basic facts concerning the Haag-Ruelle scattering theory.

6.1. Positive energy solutions of the Klein-Gordon equation.

Definition 6.1. *Let $f \in \mathcal{S}(\mathbb{R}^d)$, such that $\widehat{f}$ has compact support. The function*

$$g(t,x) = g_t(x) \ \textit{for}\ g_t = \mathrm{e}^{-\mathrm{i}t\omega(D_x)}f,$$

which solves $(\partial_t^2 - \Delta_x)g + m^2 g = 0$, will be called a positive energy KG solution.

Proposition 6.2. *There hold the following facts:*

(1) *Let $h \in C_0^\infty(\mathbb{R}^d)$. Then*

$$\mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\omega(D_x)} h\left(\frac{x}{t}\right)\mathrm{e}^{-\mathrm{i}t\omega(D_x)} = h(\nabla\omega(D_x)).$$

(2) *Let $\chi_1, \chi_2 \in C^\infty(\mathbb{R}^d)$ be bounded with all derivatives and having disjoint supports. Let $f \in \mathcal{S}(\mathbb{R}^d)$ be s.t. $\widehat{f}$ has compact support. Then*

$$\|\chi_1\left(\frac{x}{t}\right)\mathrm{e}^{-\mathrm{i}t\omega(D_x)}\chi_2(\nabla\omega(D_x))f\|_{L^2(\mathbb{R}^d)} \in O(t^{-\infty}).$$

Proof. (1) is obvious. For (2) see [RS3]. □

The following notion of *velocity support* will be useful later on.

Definition 6.3. *Let $\Delta \Subset H_m$. We set*

$$\mathrm{Vel}(\Delta) := \{\nabla\omega(p) \ : \ p \in \mathbb{R}^d, \ (\omega(p), p) \in \Delta\}.$$

Clearly if Δ_1 and Δ_2 are disjoint, then so are $\mathrm{Vel}(\Delta_1)$ and $\mathrm{Vel}(\Delta_2)$. If g is a positive energy KG solution with initial data f, then $\mathrm{supp}\widehat{g} \subset H_m$ and $\mathrm{Vel}(\mathrm{supp}\widehat{g}) = \{\nabla\omega(p) \ : \ p \in \mathrm{supp}\widehat{f}\}$ can be called the *velocity support* of g, as illustrated by Prop. 6.2 (2).

6.2. Haag-Ruelle scattering theory.

Let $B \in \mathcal{L}_0$ satisfy (2.8), i.e.,

$$-\mathrm{supp}(\widehat{B}) \cap \mathcal{S}p\, U \subset H_m.$$

Let now g be a positive energy KG solution. The *Haag-Ruelle creation operator* is given by $\{B_t^*(g_t)\}_{t\in\mathbb{R}}$, that is,

$$B_t^*(g_t) = \int g(t,x)B^*(t,x)dx.$$

Note that since $\mathrm{e}^{-\mathrm{i}t\omega(D_x)}$ preserves $\mathcal{S}(\mathbb{R}^d)$ the integral is well defined.

Lemma 6.4. *The following properties hold:*

(1) $B_t^*(g_t)\Omega = B^*(f)\Omega = (2\pi)^{d/2}\widehat{f}(P)B\Omega$, *if* $g_t = \mathrm{e}^{-\mathrm{i}t\omega(D_x)}f$.
(2) *Let $\Delta \Subset \mathbb{R}^{1+d}$, $f \in L^2(\mathbb{R}^d)$. Then $\|B^{(*)}(f)\mathbb{1}_\Delta(U)\| \le c_{\Delta,B}\|f\|_{L^2(\mathbb{R}^d)}$.*
(3) $\partial_t B_t^*(g_t) = \dot{B}_t^*(g_t) + B_t^*(\dot{g}_t)$, *where $\dot{B} = \partial_s B(s,0)_{|s=0} \in \mathcal{L}_0$ and $\dot{g} = \partial_t g$ is a positive energy KG solution with the same velocity support as g.*

Proof. We use the notation from Subsect. 3.2. We have

$$B_t^*(g_t)\Omega = \quad (\langle\overline{g_t}| \otimes \mathbb{1}) \circ a_{B_t^*}\Omega = (\langle\overline{g_t}| \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \mathrm{e}^{\mathrm{i}tH}) \circ a_{B^*}\Omega.$$

By (2.8) and (2.4) *iii)* we have $a_{B^*}\Omega = (\mathbb{1} \otimes \mathbb{1}_{H_m}(U)) \circ a_{B^*}\Omega$, hence

$$(\mathbb{1} \otimes \mathrm{e}^{\mathrm{i}tH}) \circ a_{B^*}\Omega = (\mathbb{1} \otimes \mathrm{e}^{\mathrm{i}t\omega(P)}) \circ a_{B^*}\Omega.$$

From (3.8) we obtain that:

$$(\mathbb{1} \otimes \mathrm{e}^{-\mathrm{i}y\cdot P}) \circ a_{B^*}\Omega = (\mathrm{e}^{\mathrm{i}y\cdot D_x} \otimes \mathbb{1}) \circ a_{B^*}\Omega, \ y \in \mathbb{R}^d, \tag{6.1}$$

which implies that

$$(\mathbb{1} \otimes \mathrm{e}^{\mathrm{i}t\omega(P)}) \circ a_{B^*}\Omega = (\mathrm{e}^{\mathrm{i}t\omega(D_x)} \otimes \mathbb{1}) \circ a_{B^*}\Omega,$$

using that $\omega(p) = \omega(-p)$. Hence

$$\begin{aligned} B_t^*(g_t)\Omega = & \quad (\langle\overline{g_t}| \otimes \mathbb{1}) \circ (\mathrm{e}^{\mathrm{i}t\omega(D_x)} \otimes \mathbb{1}) \circ a_{B^*}\Omega = (\langle \mathrm{e}^{-\mathrm{i}t\omega(D_x)}\overline{g_t}| \otimes \mathbb{1}) \circ a_{B^*}\Omega \\ = & \quad (\langle\overline{f}| \otimes \mathbb{1}) \circ a_{B^*}\Omega = B^*(f)\Omega. \end{aligned}$$

The fact that $B^*(f)\Omega = (2\pi)^{d/2}\widehat{f}(P)B^*\Omega$ is immediate. Statement (2) follows from Lemma 3.4, using (3.5) for B and (3.7) for B^*. In the case of B^* we also use Lemma 3.2 (1) and the fact that $\mathrm{supp}(\widehat{B})$ is compact. (3) is a trivial computation. □

The following result is a special case of the Haag-Ruelle theorem [Ha58, Ru62]. For the reader's convenience we give an elementary proof which combines ideas from [BF82, Ar99, Dy05] and exploits the bound (2) in Lemma 6.4.

Theorem 6.5. *Let $B_1, B_2 \in \mathcal{L}_0$ satisfy (2.8). Let g_1, g_2 be two positive energy KG solutions with disjoint velocity supports. Then:*

(1) *There exists the two-particle scattering state given by*

$$\Psi^+ = \lim_{t\to\infty} B_{1,t}^*(g_{1,t})B_{2,t}^*(g_{2,t})\Omega. \tag{6.2}$$

(2) *The state Ψ^+ depends only on the single-particle vectors $\Psi_i = B_{i,t}^*(g_{i,t})\Omega$, and therefore we can write $\Psi^+ = \Psi_1 \overset{\mathrm{out}}{\times} \Psi_2$. Given two such vectors Ψ^+ and $\tilde{\Psi}^+$ one has:*

$$(\tilde{\Psi}^+|\Psi^+) \quad = \quad (\tilde{\Psi}_1|\Psi_1)(\tilde{\Psi}_2|\Psi_2) + (\tilde{\Psi}_1|\Psi_2)(\tilde{\Psi}_2|\Psi_1), \tag{6.3}$$

$$U(t,x)(\Psi_1 \overset{\mathrm{out}}{\times} \Psi_2) \quad = \quad (U(t,x)\Psi_1) \overset{\mathrm{out}}{\times} (U(t,x)\Psi_2), \ (t,x) \in \mathbb{R}^{1+d}. \tag{6.4}$$

Before giving the proof of the theorem, let us explain how to obtain two-particle scattering states from arbitrary one-particle states, thereby defining the (outgoing) *two-particle wave operator*. Let

$$\mathcal{H}_m := \mathbb{1}_{H_m}(U)\mathcal{H},$$

be the space of *one-particle states*. For $\Psi_1, \Psi_2 \in \mathcal{H}$ we set

$$\Psi_1 \otimes_{\mathrm{s}} \Psi_2 := \frac{1}{\sqrt{2}}(\Psi_1 \otimes \Psi_2 + \Psi_2 \otimes \Psi_1) \in \mathcal{H} \otimes_{\mathrm{s}} \mathcal{H}.$$

Proposition 6.6. *There exists a unique isometry*

$$W_2^+ : \ \mathcal{H}_m \otimes_{\mathrm{s}} \mathcal{H}_m \to \mathcal{H}$$

with the following properties:

(1) *If Ψ_1, Ψ_2 are as in Thm. 6.5, then $W_2^+(\Psi_1 \otimes_{\mathrm{s}} \Psi_2) = \Psi_1 \overset{\mathrm{out}}{\times} \Psi_2$,*
(2) *$U(t,x) \circ W_2^+ = W_2^+ \circ (U_m(t,x) \otimes U_m(t,x))$, $(t,x) \in \mathbb{R}^{1+d}$, where we denote by $U_m(t,x)$ the restriction of $U(t,x)$ to $\mathcal{H}_m$.*

Definition 6.7. (1) *The map $W_2^+ : \mathcal{H}_m \otimes_{\mathrm{s}} \mathcal{H}_m \to \mathcal{H}$ is called the* (outgoing) two-particle wave operator.
(2) *The range of W_2^+ is denoted by $\mathcal{H}_2^+$.*

Proof of Prop. 6.6. Let us denote by $\mathcal{F} \subset \mathcal{H}_m \otimes_{\rm s} \mathcal{H}_m$ the subspace spanned by the vectors $\Psi_1 \otimes_{\rm s} \Psi_2$ for Ψ_1, Ψ_2 as in Thm. 6.5. By (6.3) there exists a unique isometry $W_2^+ : \mathcal{F} \to \mathcal{H}$ such that

$$W_2^+(\Psi_1 \otimes_{\rm s} \Psi_2) = \Psi_1 \overset{\rm out}{\times} \Psi_2,$$

for all Ψ_1, Ψ_2 as in the theorem. Moreover by (6.4) $U(t,x) \circ W_2^+ = W_2^+ \circ (U_m(t,x) \otimes U_m(t,x))$. To complete the proof of the proposition it suffices to prove that the closure of $\mathcal{F}$ is $\mathcal{H}_m \otimes_{\rm s} \mathcal{H}_m$.

Denote by (H_1, P_1), resp. (H_2, P_2) the generators of the groups $U_m(t,x) \otimes 1\!\!1$, resp. $1\!\!1 \otimes U_m(t,x)$ acting on $\mathcal{H}_m \otimes \mathcal{H}_m$, and set $(\tilde{H}, \tilde{P}) := ((H_1, P_1), (H_2, P_2))$, whose joint spectral measure is supported by $H_m \times H_m$.

By Lemma 6.4 (1) and the cyclicity of the vacuum, the set of vectors $B_t^*(g_t)\Omega$, for $B \in \mathcal{L}_0$ satisfying (2.8) and g a positive energy KG solution, is dense in $\mathcal{H}_m$. Moreover for $\Delta \Subset H_m$, the set of such vectors with g having the velocity support included in $\mathrm{Vel}(\Delta)$ is dense in $1\!\!1_\Delta(U)\mathcal{H}_m$. It follows from these density properties that the closure of $\mathcal{F}$ in $\mathcal{H}_m \otimes_{\rm s} \mathcal{H}_m$ equals

$$\mathcal{F}^{\rm cl} = \Theta_{\rm s} \circ 1\!\!1_{(H_m \times H_m) \setminus D}(\tilde{H}, \tilde{P})(\mathcal{H}_m \otimes \mathcal{H}_m),$$

where $\Theta_{\rm s} : \mathcal{H}_m \otimes \mathcal{H}_m \to \mathcal{H}_m \otimes_{\rm s} \mathcal{H}_m$ is the orthogonal projection, and $D \subset H_m \times H_m$ is the diagonal. From [BF82, Prop. 2.2] we know that the spectral measure of the restriction of (H, P) to $\mathcal{H}_m$ is absolutely continuous w.r.t. the Lorentz invariant measure on H_m. This implies that $1\!\!1_D(\tilde{H}, \tilde{P}) = 0$, which completes the proof of the proposition. $\Box$

Proof of Thm. 6.5. Let us first prove (1). Let B_1, B_2, g_1, g_2 satisfy the hypotheses of the theorem. We claim that

$$[B_{1,t}^{(*)}(g_{1,t}), B_{2,t}^{(*)}(g_{2,t})] \in O(t^{-\infty}). \tag{6.5}$$

In fact by Prop. 6.2 (2) we can find cutoff functions $\chi_1, \chi_2 \in C_0^\infty(\mathbb{R}^d)$ with disjoint supports such that

$$g_{i,t} = \chi_i\left(\frac{x}{t}\right) g_{i,t} + O(t^{-\infty}) \text{ in } L^2(\mathbb{R}^d).$$

Setting $\chi_{i,t}(x) = \chi_i(\frac{x}{t})$, this implies by Lemma 6.4 (2) that:

$$[B_{1,t}^{(*)}(g_{1,t}), B_{2,t}^{(*)}(g_{2,t})] = [B_{1,t}^{(*)}(\chi_{1,t} g_{1,t}), B_{2,t}^{(*)}(\chi_{2,t} g_{2,t})] + O(t^{-\infty}).$$

By the almost locality of $B_1^{(*)}, B_2^{(*)}$ we obtain from (3.4) and the Cauchy-Schwarz inequality that the commutator in the r.h.s. is bounded by

$$C_N \|\chi_{1,t} g_N(D_x) \chi_{2,t}\|_{\rm HS} \|g_{1,t}\|_{L^2(\mathbb{R}^d)} \|g_{2,t}\|_{L^2(\mathbb{R}^d)} \in O(t^{-\infty}),$$

which proves (6.5). (Cf. the proof of Lemma 3.10). Now we get that

$$\partial_t (B_{1,t}^*(g_{1,t})) B_{2,t}(g_{2,t}))\Omega = [\partial_t B_{1,t}^*(g_{1,t}), B_{2,t}^*(g_{2,t})]\Omega \in O(t^{-\infty}),$$

where we made use of Lemma 6.4 (1) and applied (6.5) to B_i,g_i, $\dot{B}_i$ and $\dot{g}_i$. This proves (1) by the Cook argument.

Let now $B \in \mathcal{L}_0$, satisfying (2.8), and $\Delta = -\mathrm{supp}(\widehat{B}) \cap \mathcal{S}p\, U \subset H_m$. We fix $O \subset \mathbb{R}^{1+d}$, which is an arbitrarily small neighborhood of Δ, and a function $h \in \mathcal{S}(\mathbb{R}^{1+d})$ with $\mathrm{supp} \widehat{h} \subset O$ and $\widehat{h} = (2\pi)^{-(d+1)/2}$ on Δ. Setting $C^* = \int B^*(t,x) h(t,x) dt dx$ we have: $C \in \mathcal{L}_0$ and:

$$\widehat{C^*}(E,p) = (2\pi)^{(d+1)/2} \widehat{h}(E,p) \widehat{B^*}(E,p), \ C^*\Omega = (2\pi)^{(d+1)/2} \widehat{h}(H,P) B^*\Omega.$$

This implies that $-\mathrm{supp}(\widehat{C}) \subset O$, and

$$\begin{aligned} B_t^*(g_t)\Omega &= (2\pi)^{d/2} \widehat{f}(P) B^*\Omega = (2\pi)^{d/2} \widehat{f}(P) 1\!\!1_\Delta(U) B^*\Omega \\ &= (2\pi)^{d/2} \widehat{f}(P) (2\pi)^{(d+1)/2} \widehat{h}(H,P) B^*\Omega = (2\pi)^{d/2} \widehat{f}(P) C^*\Omega = C_t^*(g_t)\Omega. \end{aligned} \tag{6.6}$$

Introducing observables C_i as above for B_i and using also (6.5) and Lemma 6.4 (2) we obtain that

$$\Psi^+ = \lim_{t\to\infty} B_{1,t}^*(g_{1,t}) B_{2,t}^*(g_{2,t})\Omega = \lim_{t\to\infty} C_{1,t}^*(g_{1,t}) C_{2,t}^*(g_{2,t})\Omega. \tag{6.7}$$

Thus we can assume that the energy-momentum transfers of B_i^* entering in the construction of scattering states are localized in arbitrarily small neighborhoods of subsets of H_m. This observation will be important in the proof of (2) to which we now proceed.

Let $\tilde{\Psi}_t = \tilde{B}^*_{1,t}(\tilde{g}_{1,t})\tilde{B}^*_{2,t}(\tilde{g}_{2,t})\Omega$ be the approximants of the scattering state $\tilde{\Psi}^+$. In order to compute the scalar product $(\tilde{\Psi}_t|\Psi_t)$ we first observe that

$$[[\tilde{B}_{1,t}(\tilde{g}_{1,t}), B^*_{1,t}(g_{1,t})], B^*_{2,t}(g_{2,t})] \in O(t^{-\infty}). \tag{6.8}$$

This relation can be justified by writing $\tilde{g}_1 = \tilde{g}_{1,1} + \tilde{g}_{1,2}$, where $\tilde{g}_{1,i}$ are positive energy KG solutions such that the velocity support of $\tilde{g}_{1,i}$ and g_i are disjoint for $i = 1, 2$. Then (6.8) follows from (6.5) and the Jacobi identity. Next we note that

$$\tilde{B}_{i,t}(\tilde{g}_{i,t})B^*_{j,t}(g_{j,t})\Omega = \Omega(\Omega|\tilde{B}_{i,t}(\tilde{g}_{i,t})B^*_{j,t}(g_{j,t})\Omega),\ 1 \le i, j \le 2. \tag{6.9}$$

This relation follows from the fact that $\tilde{B}_{i,t}(\tilde{g}_{i,t})B^*_{j,t}(g_{j,t})\Omega$ belongs to the range of $\mathbb{1}_{-K_j+\tilde{K}_i}(U)$, where K_j and $\tilde{K}_i$ are the energy-momentum transfers of B_j and $\tilde{B}_i$, respectively. In view of (6.7) $-K_j$, $-\tilde{K}_i$ can be chosen in arbitrarily small neighbourhoods of H_m. Since a non-zero vector which is a difference of two vectors from H_m is space-like, (6.9) follows.

We set for simplicity of notation $B_i(t) := B_{i,t}(g_{i,t})$, $\tilde{B}_j(t) := \tilde{B}_{j,t}(\tilde{g}_{j,t})$. Then

$$\begin{aligned}(\tilde{\Psi}_t|\Psi_t) = \ & (\Omega|\tilde{B}_2(t)B_1^*(t)\tilde{B}_1(t)B_2^*(t)\Omega)\\ & +(\Omega|\tilde{B}_2(t)B_2^*(t)\tilde{B}_1(t)B_1^*(t)\Omega)\\ & +(\Omega\ \tilde{B}_2(t)[[\tilde{B}_1(t), B_1^*(t)], B_2^*(t)]\Omega).\end{aligned} \tag{6.10}$$

Making use of (6.8) and (6.9), we conclude the proof of (6.3). It follows immediately from (6.3) that the scattering states Ψ^+ depend only on the single-particle states Ψ_i (and not on a particular choice of B_i and g_i). Finally, relation (6.4) is an easy consequence of Lemma 6.4 (1). □

7. Proof of Theorem 2.7

In the next proposition we will use the notation $N_B(h, t)$ introduced in (4.1) for $B \in \mathcal{L}_0$ and $h \in C_0^\infty(\mathbb{R}^d)$.

Proposition 7.1. *Let $i = 1, 2$, $\Delta_i \Subset H_m$ with Δ_1, Δ_2 disjoint and $B_i \in \mathcal{L}_0$ with $\mathrm{supp}(\widehat{B}_1)$, $\mathrm{supp}(\widehat{B}_2)$ disjoint. Assume moreover that:*

$$-\,\mathrm{supp}(\widehat{B}_i) \cap \mathcal{S}p\,U \subset \Delta_i, \tag{7.1}$$

$$(\Delta_i + \mathrm{supp}(\widehat{B}_i)) \cap \mathcal{S}p(U) \subset \{0\},\ i = 1, 2, \tag{7.2}$$

$$(\Delta_i + \mathrm{supp}(\widehat{B}_j)) \cap \mathcal{S}p(U) = \emptyset,\ i \ne j. \tag{7.3}$$

Let $h_i \in C_0^\infty(\mathbb{R}^d)$ with disjoint supports and $h_i \equiv 1$ on $\mathrm{Vel}(\Delta_i)$. Then for $\Psi_i \in \mathbb{1}_{\Delta_i}(U)\mathcal{H}$ one has:

$$\lim_{t\to+\infty} N_{B_1}(h_1, t)N_{B_2}(h_2, t)W_2^+(\Psi_1 \otimes_{\mathrm{s}} \Psi_2) = W_2^+(N_{B_1}(\mathbb{1})\Psi_1 \otimes_{\mathrm{s}} N_{B_2}(\mathbb{1})\Psi_2). \tag{7.4}$$

Remark 7.2. *Note that $W_2^+(\Psi_1 \otimes_{\mathrm{s}} \Psi_2)$ belongs to $\mathcal{H}_{\mathrm{c}}(U)$, and that $N_{B_i}(\mathbb{1})\Psi_i$ belong to $\mathbb{1}_{\Delta_i}(U)\mathcal{H}$, because of (7.1), (7.2), hence all the expressions appearing in (7.4) are well defined.*

Proof. We first claim that for B, Δ, Ψ, h as in the proposition one has:

$$\lim_{t\to+\infty} N_B(h, t)\Psi = N_B(\mathbb{1})\Psi. \tag{7.5}$$

In fact we first note that because of (7.1), (7.2) we have

$$B^*B\mathbb{1}_\Delta(U) = B^*|\Omega\rangle\langle\Omega|B\mathbb{1}_\Delta(U) = \mathbb{1}_\Delta(U)B^*B\mathbb{1}_\Delta(U). \tag{7.6}$$

Therefore

$$\begin{aligned} N_B(h,t)\Psi = & \quad \mathrm{e}^{\mathrm{i}tH} N_B(h_t) \mathrm{e}^{-\mathrm{i}tH}\Psi \\ = & \quad \mathrm{e}^{\mathrm{i}t\omega(P)} a_B^* \circ (\mathbb{1}_{\mathcal{H}} \otimes h_t) \circ a_B \mathrm{e}^{-\mathrm{i}t\omega(P)}\Psi \\ = & \quad a_B^* \circ \mathrm{e}^{\mathrm{i}t\omega(P+D_x)}(\mathbb{1}_{\mathcal{H}} \otimes h_t)\mathrm{e}^{-\mathrm{i}t\omega(P+D_x)} \circ a_B\Psi, \end{aligned}$$

using (3.8). Since $\mathrm{e}^{\mathrm{i}t\omega(P+D_x)} x \mathrm{e}^{-\mathrm{i}t\omega(P+D_x)} = x + t\nabla\omega(P+D_x)$, we have

$$\mathrm{e}^{\mathrm{i}t\omega(P+D_x)}(\mathbb{1}_{\mathcal{H}} \otimes h_t)\mathrm{e}^{-\mathrm{i}t\omega(P+D_x)} = h\left(\frac{x}{t} + \nabla\omega(P+D_x)\right),$$

from which we easily deduce that

$$\mathrm{s-}\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}t\omega(P+D_x)}(\mathbb{1}_{\mathcal{H}} \otimes h_t)\mathrm{e}^{-\mathrm{i}t\omega(P+D_x)} = h(\nabla\omega(P+D_x)).$$

Inserting as usual energy-momentum projections, this implies that

$$\lim_{t\to+\infty} N_B(h,t)\Psi = a_B^* \circ h(\nabla\omega(P+D_x)) \circ a_B\Psi = a_B^* a_B h(\nabla\omega(P))\Psi,$$

using once again (3.8). From the support property of h we have $h(\nabla\omega(p)) = 1$ for $(\omega(p), p) \in \Delta$, hence $h(\nabla\omega(P))\Psi = \Psi$, which completes the proof of (7.5).

We now proceed to the proof of (7.4). Since $N_{B_1}(h_1,t)N_{B_2}(h_2,t)\mathbb{1}_{\Delta_1}(U)$ is uniformly bounded in time for any $\Delta_1 \Subset \mathbb{R}^{1+d}$, it suffices by density to assume that $\Psi_i = A_{i,t}^*(g_{i,t})\Omega$ for $A_i \in \mathcal{L}_0$ satisfying (2.8) and g_i a positive energy KG solution with the velocity support included in $\mathrm{Vel}(\Delta_i)$, so that $\Psi_i = \mathbb{1}_{\Delta_i}(U)\Psi_i$. Let us fix such A_i, g_i.

By (7.3) we have $B_i A_j^*\Omega = 0$ if $i \neq j$, hence:

$$N_{B_i}(h_i,t)A_{j,t}^*(g_{j,t})\Omega = 0, \ i \neq j. \tag{7.7}$$

Next we note that for $i \neq j$:

$$\|[N_{B_i}(h_i,t), A_{j,t}^*(g_{j,t})]\| \in O(t^{-\infty}). \tag{7.8}$$

In fact since the support of h_i and the velocity support of g_j are disjoint, we can pick a smooth partition of unity $1 = \chi_i(x) + \chi_j(x)$ with $\chi_i \equiv 0$ near the velocity support of g_j and $\chi_j \equiv 0$ near the support of h_i. We have then by almost locality

$$\begin{aligned} \|[N_{B_i}(h_i,t), A_{j,t}^*(g_{j,t})]\| \leq & \quad \int \|[(B_i^* B_i)(t,x), A_j^*(t,y)]\| |h_i(\tfrac{x}{t})||g_j(t,y)|dxdy \\ & \leq C_N \int \langle x-y\rangle^{-N} |h_i(\tfrac{x}{t})||g_j(t,y)|\chi_j(\tfrac{y}{t})dxdy \\ & + C_N \int \langle x-y\rangle^{-N} |h_i(\tfrac{x}{t})||g_j(t,y)|\chi_i(\tfrac{y}{t})dxdy. \end{aligned}$$

The first integral is $O(t^{-\infty})$ because h_i and χ_j have disjoint supports, the second is also $O(t^{-\infty})$ using that $\mathrm{supp}\chi_i$ is disjoint from the velocity support of g_j and applying Prop. 6.2 (2). This proves (7.8).

Finally since $N_{B_i}(\mathbb{1})\Psi_i \in \mathbb{1}_{\Delta_i}(U)\mathcal{H}$, we can find for any $0 < \epsilon_i \ll 1$ operators $\tilde{A}_i \in \mathcal{L}_0$ and positive energy solutions $\tilde{g}_i$ satisfying the same properties as A_i, g_i such that

$$\|N_{B_i}(\mathbb{1})\Psi_i - \tilde{A}_{i,t}^*(\tilde{g}_{i,t})\Omega\| \leq \epsilon_i, \ i = 1, 2. \tag{7.9}$$

Using successively (7.8), (7.5) and (7.9), we obtain:

$$\begin{aligned} N_{B_1}(h_1,t)N_{B_2}(h_2,t)(\Psi_1 \overset{\text{out}}{\times} \Psi_2) = & \quad N_{B_1}(h_1,t)N_{B_2}(h_2,t)A_{1,t}^*(g_{1,t})A_{2,t}^*(g_{2,t})\Omega + o(t^0) \\ = & \quad N_{B_1}(h_1,t)A_{1,t}^*(g_{1,t})N_{B_2}(h_2,t)A_{2,t}^*(g_{2,t})\Omega + o(t^0) \\ = & \quad N_{B_1}(h_1,t)A_{1,t}^*(g_{1,t})N_{B_2}(\mathbb{1})\Psi_2 + o(t^0) \\ = & \quad N_{B_1}(h_1,t)A_{1,t}^*(g_{1,t})\tilde{A}_{2,t}^*(\tilde{g}_{2,t})\Omega + o(t^0) + O(t^0)\epsilon_2. \end{aligned}$$

Using then (6.5), (7.8), (7.5), we have:

$$\begin{aligned} & N_{B_1}(h_1,t)A^*_{1,t}(g_{1,t})\tilde{A}^*_{2,t}(\tilde{g}_{2,t})\Omega = N_{B_1}(h_1,t)\tilde{A}^*_{2,t}(\tilde{g}_{2,t})A^*_{1,t}(g_{1,t})\Omega + o_{\epsilon_2}(t^0) \\ = \; & \tilde{A}^*_{2,t}(\tilde{g}_{2,t})N_{B_1}(h_1,t)A^*_{1,t}(g_{1,t})\Omega + o_{\epsilon_2}(t^0) = \tilde{A}^*_{2,t}(\tilde{g}_{2,t})N_{B_1}(1\!\!1)\Psi_1 + o_{\epsilon_2}(t^0) \\ = \; & \tilde{A}^*_{2,t}(\tilde{g}_{2,t})\tilde{A}^*_{1,t}(\tilde{g}_{1,t})\Omega + o_{\epsilon_2}(t^0) + O_{\epsilon_2}(t^0)\epsilon_1 = \tilde{A}^*_{1,t}(\tilde{g}_{1,t})\tilde{A}^*_{2,t}(\tilde{g}_{2,t})\Omega + o_{\epsilon_1,\epsilon_2}(t^0) + O_{\epsilon_2}(t^0)\epsilon_1 \\ = \; & \tilde{\Psi}_1 \overset{\text{out}}{\times} \tilde{\Psi}_2 + o_{\epsilon_1,\epsilon_2}(t^0) + O_{\epsilon_2}(t^0)\epsilon_1, \end{aligned}$$

for $\tilde{\Psi}_i = \tilde{A}^*_{i,t}(\tilde{g}_{i,t})\Omega$. By Prop. 6.6 (1) we have also

$$\|N_{B_1}(1\!\!1)\Psi_1 \overset{\text{out}}{\times} N_{B_2}(1\!\!1)\Psi_2 - \tilde{\Psi}_1 \overset{\text{out}}{\times} \tilde{\Psi}_2\| \le C(\epsilon_1+\epsilon_2).$$

We obtain finally

$$\begin{aligned} & N_{B_1}(h_1,t)N_{B_2}(h_2,t)(\Psi_1 \overset{\text{out}}{\times} \Psi_2) \\ = \; & N_{B_1}(1\!\!1)\Psi_1 \overset{\text{out}}{\times} N_{B_2}(1\!\!1)\Psi_2 + o_{\epsilon_1,\epsilon_2}(t^0) + O(\epsilon_1+\epsilon_2) + O_{\epsilon_2}(t^0)\epsilon_1. \end{aligned}$$

Picking first $\epsilon_2 \ll 1$, then $\epsilon_1 \ll 1$ and then $t \gg 1$, we obtain (7.4). □

Lemma 7.3. *Let $\Delta \subset G_{2m}$ be an open bounded set. Then*

$$1\!\!1_\Delta(U)\mathcal{H}_2^+ = \mathrm{Span}\{W_2^+(\Psi_1 \otimes_{\mathrm{s}} \Psi_2) \;:\; \Psi_i \in 1\!\!1_{\Delta_i}(U)\mathcal{H},\ \Delta_i \Subset H_m,\ \Delta_1+\Delta_2 \subset \Delta,\ \Delta_1 \cap \Delta_2 = \emptyset\}^{\mathrm{cl}}.$$

Proof. The proof follows immediately from Prop. 6.6 (2) and the absolute continuity of the spectral measure of (H,P) restricted to $\mathcal{H}_m$ recalled in its proof. □

Lemma 7.4. *Let $\Delta \subset G_{2m}$ be an open bounded set s.t. $(\overline{\Delta}-\overline{\Delta}) \cap \mathcal{S}p\,U = \{0\}$. Let $\Delta_1, \Delta_2 \Subset H_m$ be disjoint and such that $\Delta_1+\Delta_2 \subset \Delta$. Then there exist $O_1, O_2 \subset \mathbb{R}^{1+d}$ which are disjoint open neighbourhoods of Δ_1, Δ_2, respectively, such that for any $K_1, K_2 \Subset \mathbb{R}^{1+d}$ satisfying $-K_i \subset O_i$, $-K_i \cap \mathcal{S}p\,U \subset \Delta_i$, one has:*

$$(\overline{\Delta}+K_1+K_2) \cap \mathcal{S}p\,U \subset \{0\}, \tag{7.10}$$

$$-(K_1+K_2) \subset \Delta, \tag{7.11}$$

$$(\Delta_i+K_i) \cap \mathcal{S}p\,U \subset \{0\}, \tag{7.12}$$

$$(\Delta_i+K_j) \cap \mathcal{S}p\,U = \emptyset, \quad i \neq j. \tag{7.13}$$

Proof. Assume that $O_i \subset \Delta_i + B(0,\varepsilon)$, where $B(0,\varepsilon)$ is the ball of radius ε centered at zero. To prove (7.10), we write

$$\begin{aligned} \overline{\Delta}+K_1+K_2 \subset \overline{\Delta}-O_1-O_2 \;&\subset\; \overline{\Delta}-\Delta_1-\Delta_2+B(0,2\varepsilon) \\ &\subset\; \overline{\Delta}-\overline{\Delta}+B(0,2\varepsilon). \end{aligned} \tag{7.14}$$

Since, by assumption, $(\overline{\Delta}-\overline{\Delta}) \cap \mathcal{S}p\,U = \{0\}$ and 0 is isolated in $\mathcal{S}p\,U$, we obtain that $(\overline{\Delta}-\overline{\Delta}+B(0,2\varepsilon)) \cap \mathcal{S}p\,U = \{0\}$ for $\varepsilon \ll 1$. As for (7.11), we obtain that

$$-(K_1+K_2) \subset O_1+O_2 \subset \Delta_1+\Delta_2+B(0,2\varepsilon) \subset \Delta, \tag{7.15}$$

for $\varepsilon \ll 1$ using that Δ_i are compact and Δ is open. Finally we write:

$$\Delta_i+K_j \subset O_i-O_j \subset \Delta_i-\Delta_j+B(0,2\epsilon). \tag{7.16}$$

We note that a difference of two vectors from H_m is either 0 or space-like. For $\varepsilon \ll 1$ we obtain (7.12) if $i=j$ and (7.13) if $i \neq j$.□

Lemma 7.5. *Let $\Delta \Subset H_m$ and $O \subset \mathbb{R}^{1+d}$ be a sufficiently small neighbourhood of Δ. Then*

$$1\!\!1_\Delta(U)\mathcal{H} = \mathrm{Span}\{\, N_B(1\!\!1)1\!\!1_\Delta(U)\mathcal{H} \;:\; B \in \mathcal{L}_0,\ -\mathrm{supp}(\widehat{B}) \subset O, -\mathrm{supp}(\widehat{B}) \cap \mathcal{S}p\,U \subset \Delta \,\}^{\mathrm{cl}}.$$

Proof. Arguing as in the proof of (7.12) we fix O sufficiently small such that for all B in the lemma one has $(\Delta + \mathrm{supp}(\widehat{B})) \cap \mathcal{S}p\, U = \{0\}$. Let now S be the subspace in the r.h.s. of the equality stated in the lemma and let P_S be the corresponding projection. By (7.6) we have $P_S \leq 1\!\!1_\Delta(U)$. To complete the proof we adapt an argument from the proof of [DT11a, Thm. 3.5]. Assume that $P_S \neq 1\!\!1_\Delta(U)$ and let $\Psi \neq 0$ with $\Psi = 1\!\!1_\Delta(U)\Psi$, $P_S\Psi = 0$. Clearly there exists $f \in \mathcal{S}(\mathbb{R}^{1+d})$ such that $\mathrm{supp}\widehat{f} \subset -O$ and $\widehat{f}(-H,-P)\Psi \neq 0$. By cyclicity of the vacuum there exists $A \in \mathfrak{A}(\mathcal{O})$, for some open bounded $\mathcal{O} \subset \mathbb{R}^{1+d}$, such that:

$$0 \neq (A^*\Omega|\widehat{f}(-H,-P)\Psi) = (\Omega|B\Psi), \text{ for } B := (2\pi)^{-\frac{1+d}{2}} \int f(t,x)A(t,x)dtdx. \tag{7.17}$$

Since $\widehat{B}(E,p) = \widehat{f}(E,p)\widehat{A}(E,p)$ we see that B satisfies the conditions from the lemma, and $B\Psi \neq 0$. By the norm continuity of $x \mapsto B(x)$ this implies that $(\Psi|N_B(1\!\!1)\Psi) \neq 0$ which contradicts the fact that $P_S\Psi = 0$. □

Proof of Thm. 2.7. In view of Thm. 2.6, it suffices to verify the inclusion

$$1\!\!1_\Delta(U)\mathcal{H}_2^+ \subset \mathrm{Span}\{\mathrm{Ran}\, Q_{2,\alpha}^+(\Delta) \ : \ \alpha \in J\}^{\mathrm{cl}}. \tag{7.18}$$

By Lemma 7.3, it is enough to show that for any $\Delta_1, \Delta_2 \Subset H_m$ such that $\Delta_1 + \Delta_2 \subset \Delta$ and $\Delta_1 \cap \Delta_2 = \emptyset$ one has

$$W_2^+\left(1\!\!1_{\Delta_1}(U)\mathcal{H} \otimes_{\mathrm{s}} 1\!\!1_{\Delta_2}(U)\mathcal{H}\right) \subset \mathrm{Span}\{\mathrm{Ran}\, Q_{2,\alpha}^+(\Delta) \ : \ \alpha \in J\}^{\mathrm{cl}}. \tag{7.19}$$

Let $O_1, O_2 \in \mathbb{R}^{1+d}$ be sufficiently small open neighbourhoods of Δ_1, Δ_2, respectively, so that the assertions of Lemma 7.4 hold. We choose $B_1, B_2 \in \mathcal{L}_0$, such that $-\mathrm{supp}(\widehat{B}_i) \subset O_i$, $-\mathrm{supp}(\widehat{B}_i) \cap \mathcal{S}p\, U \subset \Delta_i$. By Lemma 7.4, B_1, B_2 are $\Delta-$admissible in the sense of Definition 2.4 and satisfy the assumptions of Prop. 7.1. Finally, we choose $h_1, h_2 \in C_0^\infty(\mathbb{R}^d)$ as in Prop. 7.1.

Let J_0 be the set of quadruples (B_1, B_2, h_1, h_2) as specified above. We get

$$\begin{aligned} & \mathrm{Span}\{Q_{2,\alpha}^+(\Delta) \circ W_2^+\left(1\!\!1_{\Delta_1}(U)\mathcal{H} \otimes_{\mathrm{s}} 1\!\!1_{\Delta_2}(U)\mathcal{H}\right) \ : \ \alpha \in J_0\} \\ = \ & \mathrm{Span}\{W_2^+\left(N_{B_1}(1\!\!1)1\!\!1_{\Delta_1}(U)\mathcal{H} \otimes_{\mathrm{s}} N_{B_2}(1\!\!1)1\!\!1_{\Delta_2}(U)\mathcal{H}\right) \ : \ \alpha \in J_0\} \\ = \ & W_2^+\left(1\!\!1_{\Delta_1}(U)\mathcal{H} \otimes_{\mathrm{s}} 1\!\!1_{\Delta_2}(U)\mathcal{H}\right). \end{aligned} \tag{7.20}$$

In the first step we use Prop. 7.1 and in the second Lemma 7.5. Clearly, $J_0 \subset J$, thus the subspace on the l.h.s. of (7.20) is included in the subspace on the r.h.s. of (7.19). This concludes the proof. □

Appendix A. Propagation estimates for inhomogeneous evolution equations

In this section we extend standard results on propagation estimates and existence of limits for unitary propagators to the case of an inhomogeneous evolution equation:

$$\partial_t u(t) = -\mathrm{i}Hu(t) + r(t).$$

Let $\mathcal{H}$ be a Hilbert space and H a self-adjoint operator on $\mathcal{H}$. We fix a function

$$\mathbb{R}^+ \ni t \mapsto u(t) \in \mathcal{H},$$

such that

$$\begin{aligned} & i) \quad \sup_{t\geq 0} \|u(t)\| < \infty, \\ & ii) \quad u(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, \mathrm{Dom}\, H), \end{aligned} \tag{A.1}$$

and set:

$$r(t) := \partial_t u(t) + \mathrm{i}Hu(t).$$

For a map $\mathbb{R}^+ \ni t \mapsto M(t) \in B(\mathcal{H})$ we denote by $\mathcal{D}M(t) = \partial_t M(t) + [H, \mathrm{i}M(t)]$ the Heisenberg derivative of $M(t)$, w.r.t. the evolution $\mathrm{e}^{-\mathrm{i}tH}$. We assume that $[H, \mathrm{i}M(t)]$, defined first as a quadratic form on $\mathrm{Dom}\, H$, extends by continuity to a bounded operator.

The following three lemmas can be proved by mimicking standard arguments, see e.g. [DG97, Sect. B.4]. By $C_j(\,\cdot\,)$, $B(\,\cdot\,)$, $B_1(\,\cdot\,)$ we denote auxiliary functions from $\mathbb{R}^+$ to $B(\mathcal{H})$.

Lemma A.1. *Let* $\mathbb{R}^+ \ni t \mapsto M(t) \in B(\mathcal{H})$ *be such that:*

$$i) \quad \sup_{t\in\mathbb{R}^+} \|M(t)\| < \infty,\ \|M(\cdot)r(\cdot)\|,\ \|M^*(\cdot)r(\cdot)\| \in L^1(\mathbb{R}^+, dt),$$

$$ii) \quad \mathcal{D}M(t) \geq B^*(t)B(t) - \sum_{j=1}^n C_j^*(t)C_j(t),\ \int_{\mathbb{R}^+} \|C_j(t)u(t)\|^2 dt < \infty.$$

Then

$$\int_0^{+\infty} \|B(t)u(t)\|^2 dt < \infty.$$

Lemma A.2. *Let* $\mathbb{R}^+ \ni t \mapsto M(t) \in B(\mathcal{H})$ *be such that:*

$$i) \quad \sup_{t\in\mathbb{R}^+} \|M(t)\| < \infty,\ \|M(\cdot)r(\cdot)\|,\ \|M^*(\cdot)r(\cdot)\| \in L^1(\mathbb{R}^+, dt),$$

$$ii) \quad |(u_1|\mathcal{D}M(t)u_2)| \leq \sum_{j=1}^n \|C_j(t)u_1\|\|C_j(t)u_2\|,\ u_1, u_2 \in \mathcal{H},$$

$$\textit{with} \quad \int_{\mathbb{R}^+} \|C_j(t)u(t)\|^2 dt < \infty.$$

Then

$$\lim_{t\to+\infty} (u(t)|M(t)u(t)) \ \textit{exists.}$$

Lemma A.3. *Let* $\mathbb{R}^+ \ni t \mapsto M(t) \in B(\mathcal{H})$ *be such that:*

$$i) \quad \|M(\cdot)r(\cdot)\| \in L^1(\mathbb{R}^+, dt),$$

$$ii) \quad |(u_1|\mathcal{D}M(t)u(t))| \leq \|B_1(t)u_1\|\|B(t)u(t)\|,\ \textit{with}$$

$$iii) \quad \int_{\mathbb{R}^+} \|B(t)u(t)\|^2 dt < \infty,\ \int_{\mathbb{R}^+} \|B_1(t)\mathrm{e}^{-\mathrm{i}tH}u_1\|^2 dt \leq C\|u_1\|^2,\ u_1 \in \mathcal{H}.$$

Then

$$\lim_{t\to+\infty} \mathrm{e}^{\mathrm{i}tH} M(t)u(t) \ \textit{exists.}$$

References

[AH67] H. Araki and R. Haag: *Collision cross sections in terms of local observables.* Commun. Math. Phys. **4**, (1967) 77–91.

[Ar99] H. Araki: *Mathematical theory of quantum fields.* Oxford Science Publications, 1999.

[Ar74] W. Arveson: *On groups of automorphisms of operator algebras.* J. Funct. Anal. **15**, (1974) 217–243.

[Ar82] W. Arveson: *The harmonic analysis of automorphism groups.* In Operator algebras and applications, Part I (Kingston, Ont., 1980), Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I.,1982.D., pp. 199–269.

[Bu90] D. Buchholz: *Harmonic analysis of local operators.* Commun. Math. Phys. **129**, (1990) 631–641.

[BF82] D. Buchholz and K. Fredenhagen: *Locality and the structure of particle states.* Commun. Math. Phys. **84**, (1982) 1–54.

[BPS91] D. Buchholz, M. Porrmann and U. Stein: *Dirac versus Wigner: Towards a universal particle concept in quantum field theory.* Phys. Lett. B **267**, (1991) 377–381.

[CD82] M. Combescure and F. Dunlop: *Three-body asymptotic completeness for $P(\phi)_2$ models.* Commun. Math. Phys. **85**, (1982) 381–418.

[Dy05] W. Dybalski: *Haag-Ruelle scattering theory in presence of massless particles.* Lett. Math. Phys. **72**, (2005) 27–38.

[Dy10] W. Dybalski: *Continuous spectrum of automorphism groups and the infraparticle problem.* Commun. Math. Phys. **300**, (2010) 273–299.

[DM12] W. Dybalski and J.S. Møller: *The translation invariant massive Nelson model: III. Asymptotic completeness below the two-boson threshold.* Preprint arXiv:1210.6645 [math-ph].

[DT11a] W. Dybalski and Y. Tanimoto: *Asymptotic completeness for infraparticles in two-dimensional conformal field theory.* Preprint arXiv1112.4102 [math-ph].

[DT11b] W. Dybalski and Y. Tanimoto: *Infraparticles with superselected direction of motion in two-dimensional conformal field theory.* Commun. Math. Phys. **311**, (2012) 457–490.

[De93] J. Dereziński: *Asymptotic completeness of long-range N-body quantum systems.* Ann. of Math. **138**, (1993) 427–476.

[DG99] J. Dereziński and C. Gérard: *Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians.* Rev. Math. Phys. **11**, (1999) 383–450.

[DG97] J. Dereziński and C. Gérard: *Scattering theory of classical and quantum N-particle systems.* Springer, 1997.

[DG00] J. Dereziński and C. Gérard: *Spectral and scattering theory of spatially cut-off $P(\phi)_2$ Hamiltonians.* Commun. Math. Phys. **213**, (2000) 39–125.

[En75] V. Enss: *Characterization of particles by means of local observables.* Commun. Math. Phys. **45**, (1975) 35–52.

[En78] V. Enss: *Asymptotic completeness for quantum mechanical potential scattering.* Commun. Math. Phys. **61**, (1978) 285–291.

[FGS02] J. Fröhlich, M. Griesemer and B. Schlein: *Asymptotic completeness for Rayleigh scattering.* Ann. Henri Poincaré **3**, (2002) 107–170.

[FGS04] J. Fröhlich, M. Griesemer and B. Schlein: *Asymptotic completeness for Compton scattering.* Commun. Math. Phys. **252**, (2004) 415–476.

[GJS73] J. Glimm, A. Jaffe and T. Spencer: *The particle structure of the weakly coupled $P(\phi)_2$ model and other applications of high temperature expansions: Part I. Physics of quantum field models. Part II. The cluster expansion.* In: Constructive quantum field theory. (Erice, 1973), G. Velo, A. S. Wightman (eds.). Berlin, Heidelberg, New York: Springer 1973.

[Ge91] C. Gérard: *Mourre estimate for regular dispersive systems*, Ann. Inst. H. Poincaré **54**, (1991) 59-88.

[Gr90] G. M. Graf: *Asymptotic completeness for N-body short-range quantum systems: a new proof.* Commun. Math. Phys. **132**, (1990) 73–101.

[Ha58] R. Haag: *Quantum field theories with composite particles and asymptotic conditions.* Phys. Rev. **112**, (1958) 669–673.

[Ha] R. Haag: *Local quantum physics.* Springer, 1992.

[Po04a] M. Porrmann: *Particle weights and their disintegration I.* Commun. Math. Phys. **248**, (2004) 269–304.

[Po04b] M. Porrmann: *Particle weights and their disintegration II.* Commun. Math. Phys. **248**, (2004) 305–333.

[Ru62] D. Ruelle: *On the asymptotic condition in quantum field theory.* Helv. Phys. Acta **35**, (1962) 147–163.

[RS3] M. Reed and B. Simon: *Methods of modern mathematical physics. Part III: Scattering theory.* Academic Press, 1979.

[SiSo87] I. M. Sigal and A. Soffer: *The N-particle scattering problem: asymptotic completeness for short-range systems.* Ann. of Math. **126**, (1987) 35–108.

[SZ76] T. Spencer and F. Zirilli: *Scattering states and bound states in $\lambda P(\phi)_2$.* Commun. Math. Phys. **49**, (1976) 1–16.

[Zi97] L. Zieliński: *Scattering for a dispersive charge-transfer model.* Ann. Inst. Henri Poincaré **67**, (1997) 339–386.

Zentrum Mathematik, Technische Universität München, D-85747 Garching Germany
E-mail address: `dybalski@ma.tum.de`

Département de Mathématiques, Université de Paris XI, 91405 Orsay Cedex France
E-mail address: `christian.gerard@math.u-psud.fr`